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THE EDSAC

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The EDSAC (Electronic Delay Storage Automatic Calculator)

1. Introduction. The EDSAC is a serial electronic calculating machine now in operation in the University Mathematical Laboratory, Cambridge, England. It works in the scale of two and uses ultrasonic tanks for storage. The main store consists of 32 tanks each of which is about 5 ft. long and holds 32 numbers of 17 binary digits, one being a sign digit. This gives 1,024 storage locations in all. It is possible to combine two adjacent storage locations so as to accommodate a number with 35 binary digits (including a sign digit); thus at any time the store may contain a mixture of *long* and *short* numbers. Short tanks which can hold one number only are used for the accumulator and multiplier registers in the arithmetical unit, and for control purposes in various parts of the machine.

A single address code is used in the EDSAC, orders being of the same length as short numbers. Most orders consist of a functional part which defines the operation and a numerical part which defines a storage location; some orders, however, consist of a functional part only. The complete order code is as follows.

EDSAC Order Code

Order	Explanation
$A\ n$	Add the number in storage location n into the accumulator.
$S\ n$	Subtract the number in storage location n from the accumulator.
$H\ n$	Transfer the number in storage location n into the multiplier register.
$V\ n$	Multiply the number in storage location n by the number in the multiplier register and add into the accumulator.
$N\ n$	Multiply the number in storage location n by the number in the multiplier register and subtract from the contents of the accumulator.
$T\ n$	Transfer the contents of the accumulator to storage location n, and clear the accumulator.
$U\ n$	Transfer the contents of the accumulator to storage location n, and do not clear the accumulator.
$C\ n$	Collate the number in storage location n with the number in the multiplier register, i.e., add a 1 into the accumulator in digital positions where both numbers have a 1 and a 0 in other digital positions.
$R\ 2^{n-2}$	Shift the number in the accumulator n places to the right, i.e., multiply it by 2^{-n}.
$L\ 2^{n-2}$	Shift the number in the accumulator n places to the left, i.e., multiply it by 2^{n}.
$E\ n$	If the number in the accumulator is greater than or equal to zero, execute next the order which stands in storage location n; otherwise, proceed serially.
$G\ n$	If the number in the accumulator is less than zero, execute next the order which stands in storage location n; otherwise, proceed serially.
$I\ n$	Read the next row of holes on the tape, and place the resulting 5 digits in the least significant places of storage location n.
$O\ n$	Print the character now set up on the teleprinter, and set up on the teleprinter the character represented by the five most significant digits in storage location n.
$F\ n$	Place the five digits which represent the character next to be printed by the teleprinter in the five most significant places of storage location n.
Y	Round off the number in the accumulator to 34 binary digits.
Z	Stop the machine, and ring the warning bell.

Ordinary 5-hole telegraphic punched tape is used for input. Each row of holes represents a 5-digit binary number and the basic input operation is to transfer this number to the store. Similarly, the output mechanism is a teleprinter, and the basic output operation is to transfer a 5-digit binary number to the printer, and to print the corresponding character. The teleprinter code is chosen so that binary numbers up to nine are printed as the corresponding figures and a similar code is used for input. This enables the operation of conversion to and from the decimal system to be programmed as part of the calculation.

The purpose of the *F* order is to enable the operation of the printer to be checked. Apart from this, no special checking facilities are provided in the EDSAC, and it is left to the programmer to incorporate in the program such checks as he considers necessary.

2. The Control Sequence. When the machine is in operation, orders are executed automatically in the order in which they stand in the store. However, when a conditional order (*E* or *G*) is encountered and the condition is satisfied, the next order to be executed is the one which stands in the storage location specified in the conditional order.

Count is kept of the orders as they are executed by means of a short tank—known as the sequence control tank—which has associated with it an adding circuit through which unity is added to the number stored in the tank each time an order is executed. During a conditional order (when the condition is satisfied) unity is not added, but, instead, the number in the tank is replaced by the numerical part of the conditional order.

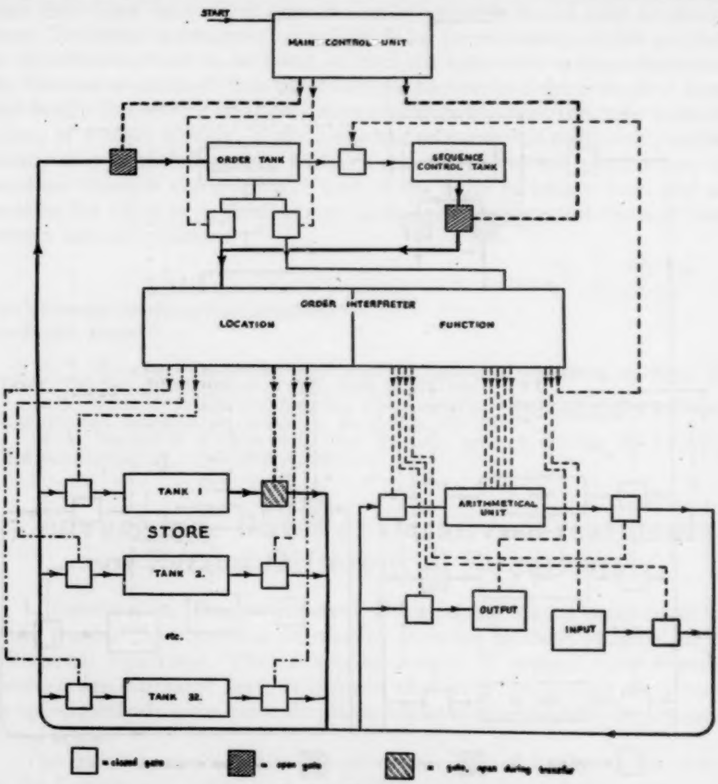
The control sequence of the machine falls into two parts. In stage I, an order is transferred from the location in the store given by the number in the sequence control tank to a short tank known as the order tank. In stage II the order in the order tank is executed.

The various constituent units of the machine—store, arithmetic unit, input unit, output unit, sequence control tank, and order tank—, are connected together through gates, so that the interconnections proper to each successive part of the control sequence can be made by opening the appropriate gates. The gates are operated by waveforms supplied either by the main control unit or by a part of the machine called, for the purpose of this description, "order interpreter." In the machine itself the "order interpreter" consists of a number of separate units. In the diagrams that follow, wires which carry pulses are shown as continuous lines, while those which carry control waveforms are shown as dotted lines.

Figure 1 shows the state of the machine during stage I of the control sequence. Here it is assumed that the storage location specified by the number in the sequence control tank is in tank 1 of the main store. The wire along which the order passes from this storage location to the order tank is shown by a heavy line in the diagram, and similarly the wires along which the control waveforms pass are shown as heavy dotted lines. The location section of the "order interpreter" receives the number in the sequence control tank and emits a waveform which opens the output gate of Tank 1. There are 32 orders (or short numbers, or a mixture of orders, short numbers, and long numbers) circulating in the tank. The gate is opened as soon as the required number becomes available and closed again when it has passed through. In this respect it is treated differently from the other gates in the machine

which remain open or closed during the whole of stage I; The difference is indicated in the diagram by the different form of shading used. The switching and timing necessary for the operations detailed above are carried out within the "order interpreter."

Stage II is very similar to stage I except that the type of operation performed is not restricted to a simple transfer from the store to the order tank. During stage II, the order tank is connected to the "order interpreter"



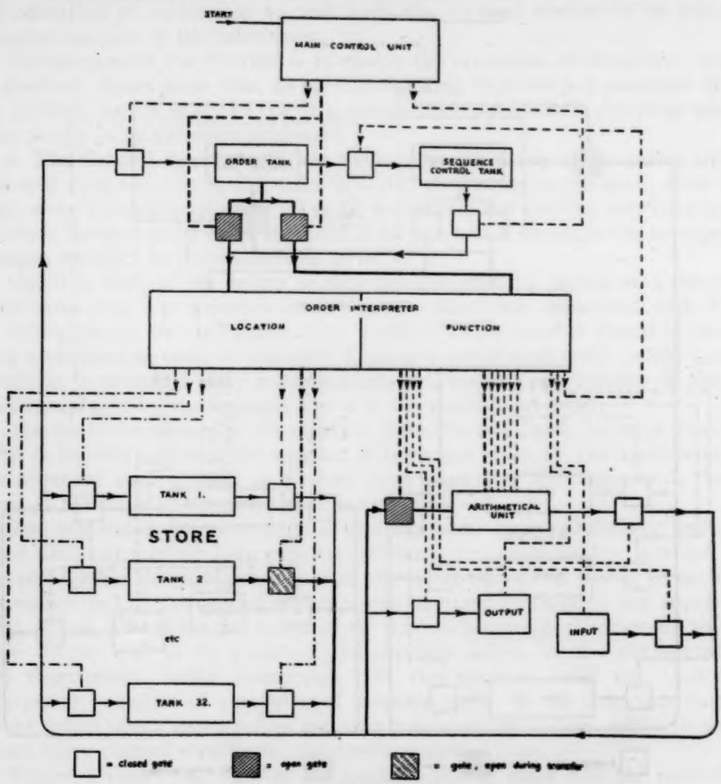
STAGE I

FIG. 1

in place of the sequence control tank, and the numerical part of the order is dealt with in the same way as the number in the sequence control tank during stage I. A new feature, however, is that the function section of the "order interpreter" comes into action and interprets the functional part of the order; it proceeds to emit waveforms which (a) set up connections between the units ready for the passage of numbers and (b) place the arithmetical unit (or input or output units, as the case may be) in a state of readiness for executing the order. Figure 2 shows the various gates set up for the

execution of an add order calling for the transfer of a number from tank 2 of the main store to the arithmetical unit. One of the control wires passing from the "order interpreter" to the arithmetical unit is shown as a heavy (dotted) line to indicate that the arithmetical unit has been prepared for an addition.

The above description has been written with the case of an arithmetical order involving the store in mind. In the case of other orders (e.g., left or



STAGE 2

FIG. 2

right shift, or a conditional order), stage II is simplified in that the location section of the "order interpreter" is not used. For example, during stage II for a conditional order, the order gate is connected to the sequence control tank through the gate shown in the diagram and the whole order transferred from one tank to the other. When an order is in the sequence control tank, only the numerical part is used, the function part being irrelevant.

3. Initial Orders. From what has been said, and from an examination of the order code, it will be seen that the input mechanism is controlled by

program orders. Unless, therefore, there are some orders in the store at the beginning of the computation, nothing can be taken in through the input, and the machine cannot start. For this reason, there is a sequence of orders, known as initial orders, permanently wired onto a set of uniselectors (rotary telephone switches). These orders can be transferred to the store by pressing a button.

There is considerable latitude in the choice of the initial orders, although once they have been wired onto the uniselectors, it is not easy to change them. The initial orders used in the EDSAC at present enable orders punched in the following form to be taken in from the tape. First a letter indicating the function is punched, then the numerical part of the order in decimal form, and finally the letter F or D indicating, respectively, that the order refers to a long or a short number. If the order has no numerical part, it is punched simply as a letter followed by F. Under the control of the initial orders the machine converts the numerical part of the order to binary form and assembles the order with the function digits and the numerical digits in their correct relative positions.

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Convergence Rates of Iterative Treatments of Partial Differential Equations

1. Introduction. The development of high-speed digital computers[1] has made feasible the numerical solution by iterative methods of some partial differential equations. The convergence rates of several such iterative methods are estimated here. It is found that with the familiar elementary iterative methods some quite simple problems require prohibitive computational labor.

The iterative methods here considered are related to the various forms of the Southwell "relaxation method"[2,4] in that they involve successively applied local corrections to improve an approximate solution. However, these iterative methods are routinized in conformity with the requirements of automatic computers while the relaxation method is flexible and depends in an essential way on the skill of its practitioners.

2. Reduction to Finite Difference Form. The iterative methods of successive approximation considered here are, like the relaxation method, not directly applicable to partial differential equations (and associated boundary conditions) but only to the finite difference approximations to them derived in the customary way.[3] For example, the Laplace equations, $\Delta\phi = 0$, applicable within a region $\mathcal{R}$ in the x, y plane may be approximated by the

difference equation:

$$(1)\quad L\phi(x, y) \equiv \phi(x+h, y) + \phi(x-h, y) + \phi(x, y+h) + \phi(x, y-h) - 4\phi(x, y) = 0$$

applied at those points $(x = jh, y = kh)$, of a rectangular lattice which lie within the region $\mathcal{R}$. We are here concerned only with the rate of convergence to the solution of this set of algebraic equations and not with closeness of that approximation to the solution of the differential equation. In the following we will use this difference equation, applied within a rectangular region, as an illustrative example for each of the iteration methods considered. We denote the value assigned to $\phi(jh, kh)$ in the nth stage of iteration by $\phi^n_{j,k}$. The limit approached with increasing n we denote by $\phi_{j,k}$. For definiteness we take $\mathcal{R}$ to be the rectangular region

$$\mathcal{R} = \{j = 0, 1, \cdots, p, \quad k = 0, 1, \cdots, q\};$$

on the boundary of this region we assign fixed values of ϕ. Thus

$$\phi^n_{j,k} = \phi_{j,k} = b_{j,k} \qquad \text{for} \qquad j = 0 \text{ or } p, \qquad k = 0 \text{ or } q.$$

$$(2)\quad L\phi_{j,k} \equiv \phi_{j-1,k} + \phi_{j+1,k} + \phi_{j,k-1} + \phi_{j,k+1} - 4\phi_{j,k} = 0 \quad \text{for} \quad j = 1, 2, \cdots, p-1, \quad k = 1, 2, \cdots, q-1.$$

In all of the methods here considered $\phi^0_{j,k}$ is a first (guessed) approximation to $\phi_{j,k}$. Each of the succeeding approximations, $\phi^1, \phi^2, \cdots$, is calculated on the basis of its predecessors (or immediate predecessor) by some process which guarantees the convergence of ϕ^n to ϕ.

3. The Richardson Method. Following SHORTLEY, WELLER and FRIED[4] we term the most elementary of the familiar iterative procedures the RICHARDSON method.[5] In this method the correction process applied to each ϕ^n consists of the addition of a positive multiple of $L\phi^n$ (for each interior point). Thus

$$(3)\quad \begin{aligned} \phi^{n+1}_{j,k} &= \phi^n_{j,k} + \alpha L\phi^n_{j,k} \equiv \phi^n_{j,k} \\ &\quad + \alpha[\phi^n_{j-1,k} + \phi^n_{j+1,k} + \phi^n_{j,k-1} + \phi^n_{j,k+1} - 4\phi^n_{j,k}] \quad \text{(interior points)} \\ \phi^{n+1}_{j,k} &= \phi^n_{j,k} = b_{j,k} \quad \text{(boundary points)}. \end{aligned}$$

The error at each stage we denote by $\epsilon^n_{j,k}$,

$$\epsilon^n_{j,k} \equiv \phi^n_{j,k} - \phi_{j,k}.$$

By substitution in (3) we obtain the error recurrence relation,

$$\epsilon^{n+1}_{j,k} = \epsilon^n_{j,k} + \alpha L\epsilon^n_{j,k} \quad \text{(interior points)},$$

since $L\phi_{j,k} = 0$; or more briefly

$$(4)\quad \begin{aligned} \epsilon^{n+1} &= (1 + \alpha L)\epsilon^n \quad \text{(interior points)} \\ &= 0 \quad \text{(boundary points)}. \end{aligned}$$

The most familiar form of the Richardson method is that obtained by setting $\alpha = \frac{1}{4}$. Then equation (3) reduces to

$$(5)\quad \begin{aligned} \phi^{n+1}_{j,k} &= \tfrac{1}{4}[\phi^n_{j-1,k} + \phi^n_{j+1,k} + \phi^n_{j,k-1} + \phi^n_{j,k+1}] \quad \text{(interior points)} \\ &= b_{j,k} \quad \text{(boundary points)}. \end{aligned}$$

As shown below, this form is not only numerically more convenient, by reason of the disappearance of $\phi^n_{j,k}$ from the right member, but is also in one sense the most efficient form.

To determine the convergence rate of this process we expand ϵ^0 in the eigenfunctions of the operator L subject to the boundary condition of (4). These are evidently,[6]

$$\epsilon_{j,k}^{(r,s)} = \sin(\pi r j/p)\sin(\pi s k/q), \qquad \begin{array}{l} r = 1, 2, \cdots, p-1 \\ s = 1, 2, \cdots, q-1. \end{array} \tag{6}$$

The corresponding eigenvalue of L we denote by $L_{(r,s)}$

$$\begin{aligned} L\epsilon_{j,k}^{(r,s)} &= \epsilon_{j-1,k}^{(r,s)} + \epsilon_{j+1,k}^{(r,s)} + \epsilon_{j,k-1}^{(r,s)} + \epsilon_{j,k+1}^{(r,s)} - 4\epsilon_{j,k}^{(r,s)} \\ &= [\sin(\pi r(j-1)/p) + \sin(\pi r(j+1)/p) - 2\sin(\pi r j/p)]\sin(\pi s k/q) \\ &\quad + [\sin(\pi s(k-1)/q) + \sin(\pi s(k+1)/q) - 2\sin(\pi s k/q)]\sin(\pi r j/p) \\ &= [2\cos(\pi r/p) - 2]\sin(\pi r j/p)\sin(\pi s k/q) \\ &\quad + [2\cos(\pi s/q) - 2]\sin(\pi r j/p)\sin(\pi s k/q) \\ &= [2\cos(\pi r/p) + 2\cos(\pi s/q) - 4]\epsilon_{j,k}^{(r,s)} = L_{(r,s)}\epsilon_{j,k}^{(r,s)}, \end{aligned} \tag{7}$$

hence

$$L_{(r,s)} = -4[\sin^2(\pi r/2p) + \sin^2(\pi s/2q)].$$

All of these eigenvalues are negative. The smallest and largest in magnitude belong to $r = s = 1$ and to $r = p - 1$, $s = q - 1$, respectively. We denote these by L_0 and L_m.

$$\begin{aligned} L_0 &= -4[\sin^2(\pi/2p) + \sin^2(\pi/2q)] \cong -\pi^2(p^{-2} + q^{-2}) \\ L_m &= -4[\sin^2(\pi(p-1)/2p) \\ &\qquad + \sin^2(\pi(q-1)/2q)] \cong -8 + \pi^2(p^{-2} + q^{-2}). \end{aligned} \tag{8}$$

The eigenfunctions of L are also eigenfunctions of the iteration operation, $K \equiv (1 + \alpha L)$, corresponding to the eigenvalues

$$K_{(r,s)} = 1 + \alpha L_{(r,s)}. \tag{9}$$

Each error-eigenfunction component is multiplied by its $K_{(r,s)}$ at each iteration. Thus if

$$\epsilon_{j,k}^{0} = \sum_{r,s} a_{r,s}\epsilon_{j,k}^{(r,s)},$$

then

$$\epsilon_{j,k}^{n} = \sum_{r,s} a_{r,s}\epsilon_{j,k}^{(r,s)}K_{(r,s)}^{n}. \tag{10}$$

With increasing n those components, $\epsilon^{(r,s)}$, for which

$$|K_{(r,s)}| < 1 \tag{11}$$

diminish in importance, the more rapidly the smaller is $|K_{(r,s)}|$. This iteration method is convergent (in general) only if (11) is satisfied for all r, s. The ultimate convergence rate is determined by the maximum magnitude of K,

$$K^* \equiv \max_{(r,s)} |K_{(r,s)}|. \tag{12}$$

Since by (9) $K_{(r,s)}$ lies in the range

$$K_m \equiv 1 + \alpha L_m \leqslant K_{(r,s)} \leqslant 1 + \alpha L_0 \equiv K_0 \tag{13}$$

the equality signs holding for $(r, s) = (1, 1)$ and $(p - 1, q - 1)$, K is determined only by these extremes,

$$K^* = \max\{|K_m|, \quad |K_0|\}.$$

As α increases from zero, K_0 drops slowly from unity, K_m drops rapidly from unity. Thus $K^* = K_0 > 0$ so long as $K_0 \geqslant -K_m$. For greater values of α, $K^* = -K_m$, hence K^* then rises with increasing α. The smallest K^* (hence the most rapid convergence) occurs where

$$K_0 = 1 + \alpha L_0 = -K_m = -(1 + \alpha L_m),$$

hence for

$$\alpha = -2/(L_0 + L_m) = \tfrac{1}{4}. \tag{14}$$

For this optimum α-value the error eigenfunctions of longest and shortest "wavelength," $\epsilon^{(1,1)}$ and $\epsilon^{(p-1,q-1)}$, decay at the same rate

$$\begin{aligned} K^* &= 1 - \sin^2(\pi/2p) - \sin^2(\pi/2q) \\ &= \tfrac{1}{2}[\cos(\pi/p) + \cos(\pi/q)] \cong 1 - \pi^2(p^{-2} + q^{-2})/4 \end{aligned} \tag{15}$$

(the short wavelength error alternating in sign) while other errors decay more rapidly.

The numerical processes carried out in this iterative approach to the solution of a Laplace equation may also be regarded as the successive steps in the solution of the heat flow equation,

$$\begin{aligned} \frac{\partial \phi}{\partial t} &= \Delta\phi \quad \text{(interior)} \\ \phi &= b \quad \text{(boundary).} \end{aligned}$$

The parameter, α, then plays the rôle of

$$\frac{\Delta t}{(\Delta x)^2} = \frac{\Delta t}{(\Delta y)^2} = \frac{\Delta t}{h^2}.$$

This method of solution of the heat flow equation is unstable[7] for α appreciably greater than $\frac{1}{4}$, i.e., for a time interval appreciably greater than $\frac{1}{4}(\Delta x)^2 \equiv \frac{1}{4}(\Delta y)^2$. For smaller α-values the solution approaches asymptotically a stationary form, hence one satisfying the Laplace equation.

The optimum property of $\alpha = \frac{1}{4}$ in the Richardson treatment of the Laplace equation is not peculiar to the boundary conditions here considered. If $(L + 4)$ has an eigenfunction, ϵ'_{jk}, belonging to the eigenvalue $(L' + 4)$,

$$(L' + 4)\epsilon'_{jk} = \epsilon'_{j-1,k} + \epsilon'_{j+1,k} + \epsilon'_{j,k-1} + \epsilon'_{j,k+1},$$

then $\epsilon'' \equiv (-1)^{j+k}\epsilon'$, if it is consistent with the boundary conditions, is an eigenfunction belonging to the eigenvalue $(L'' + 4) = -(L' + 4)$,

$$\begin{aligned} (L'' + 4)\epsilon''_{j,k} = -(L' + 4)\epsilon''_{j,k} &= \epsilon''_{j-1,k} + \epsilon''_{j+1,k} + \epsilon''_{j,k-1} + \epsilon''_{j,k+1} \\ L'' &= 8 - L'. \end{aligned}$$

If the boundary conditions permit this reversal, $\epsilon' \to \epsilon''$, then for each eigenvalue, L_i, there also occurs the eigenvalue $-8 - L_i$. Then $L_0 + L_m = -8$ leading to optimum convergence for $\alpha = \frac{1}{4}$.

For other linear equations,

$$L\phi = 0,$$

where L is the linear difference operator corresponding to the linear differential operator under consideration, the Richardson method,

$$\phi^{n+1} = \phi^n + \alpha L\phi^n, \tag{16}$$

may be made convergent provided all of the eigenvalues of L are of the same sign (and not zero).

The optimum convergence rate is given by

$$\begin{aligned} \alpha &= -\,2/(L_0 + L_m) \\ K^* &= 1 - \alpha L_0 = 1 - 2\eta, \end{aligned} \tag{17}$$

where η denotes the ratio,

$$\eta = L_0/(L_0 + L_m). \tag{18}$$

Here again the Richardson method is formally equivalent to the solution of a partial differential equation in one more variable,

$$\frac{\partial \phi}{\partial t} = L\phi,$$

the solution being carried to a sufficiently great t to make the rate of change with t negligible, hence also

$$L\phi \cong 0.$$

4. The Liebmann Method. In the LIEBMANN method[8] a correction process like that of the Richardson method is applied to each of the lattice points in succession in a regular pattern. The ϕ-value so corrected is used in all subsequent operations in that iteration step. It may thus be termed a "continuous substitution method." In its simplest form the lattice is scanned in the same direction along successive rows. Thus, as applied to the Laplace equation and boundary conditions described above, the Liebmann iteration process may be written,

$$\begin{aligned} \phi^{n+1}_{j,k} &= \phi^n_{j,k} + \alpha[\phi^{n+1}_{j-1,k} + \phi^n_{j+1,k} + \phi^{n+1}_{j,k-1} + \phi^n_{j,k+1} - 4\phi^n_{j,k}] && \text{(interior points)} \\ &= b_{j,k} && \text{(boundary points).} \end{aligned} \tag{19}$$

If α is again given the value $\frac{1}{4}$ this expression becomes

$$\phi^{n+1}_{j,k} = \tfrac{1}{4}[\phi^{n+1}_{j-1,k} + \phi^n_{j+1,k} + \phi^{n+1}_{j,k-1} + \phi^n_{j,k+1}], \tag{20}$$

thus $L\phi_{j,k}$ is brought to zero (momentarily) at each of the lattice points in succession. In this form the Liebmann procedure may be regarded as a very mechanical application of the relaxation method.

The iteration equation for the error

$$\epsilon^n_{j,k} \equiv \phi^n_{j,k} - \phi_{j,k}$$

is obtained by substitution in (19),

$$\begin{aligned} \epsilon^{n+1}_{j,k} &= \epsilon^n_{j,k} + \alpha[\epsilon^{n+1}_{j-1,k} + \epsilon^n_{j+1,k} + \epsilon^{n+1}_{j,k-1} + \epsilon^n_{j,k+1} - 4\epsilon^n_{j,k}] && \text{(interior points)} \\ &= 0 && \text{(boundary points).} \end{aligned} \tag{21}$$

This error interation process can be written briefly as

$$\epsilon^{n+1} = K(\alpha)\epsilon^{n}, \tag{22}$$

where $K(\alpha)$ is a linear operator depending on the parameter α, but now not simply related to the Laplace (difference) operator, L. We again examine the spectrum of eigenvalues of $K(\alpha)$ and regard the greatest magnitude of these eigenvalues as a measure of convergence rate. If K is an eigenvalue of (22) then its eigenfunction $\epsilon_{j,k}$ (we suppress indexing of the eigenfunctions and eigenvalues) must satisfy the following equation, obtained by substituting (22) in (21)

$$\begin{aligned} K\epsilon_{j,k} &= \epsilon_{j,k} + \alpha[K\epsilon_{j-1,k} + \epsilon_{j+1,k} + K\epsilon_{j,k-1} + \epsilon_{j,k+1} - 4\epsilon_{j,k}], \\ \epsilon_{0,k} &= \epsilon_{p,k} = \epsilon_{j,0} = \epsilon_{j,q} = 0. \end{aligned} \tag{23}$$

We seek solutions of the form

$$\epsilon_{j,k} = A^j \sin(\pi rj/p) \cdot B^k \sin(\pi sk/q) \qquad \begin{aligned} r &= 1, 2, \cdots, p-1 \\ s &= 1, 2, \cdots, q-1. \end{aligned} \tag{24}$$

Substituting in (23) gives,

$$\begin{aligned} (K - 1 + 4\alpha)A^jB^k \sin(\pi rj/p)\sin(\pi sk/q) \\ = B^k \sin(\pi sk/q)[KA^{j-1}\sin(\pi r(j-1)/p) \\ + A^{j+1}\sin(\pi r(j+1)/p)] + \alpha A^j \sin(\pi rj/p) \\ \times [KB^{k-1}\sin(\pi s(k-1)/q) + B^{k+1}\sin(\pi s(k+1)/q)]. \end{aligned} \tag{25}$$

To prevent the appearance of terms in $\cos(\pi rj/p)$ and $\cos(\pi sk/q)$ in the right member of (25) we must require

$$A^2 = B^2 = K. \tag{26}$$

Since multiplication of A (or B) by -1 is equivalent to the replacement of r by $p - r$ (or s by $q - s$) we may, without loss of generality, take $A = B$. Equation (25) then becomes

$$(A^2 - 1 + 4\alpha) = 2\alpha A[\cos(\pi r/p) + \cos(\pi s/q)]$$

or

$$A^2 - 2\alpha tA + (4\alpha - 1) = 0; \quad t \equiv [\cos(\pi r/p) + \cos(\pi s/q)]. \tag{27}$$

The appearance of two values (in general) of A for each (r, s) would seem to give more than the $(p-1)(q-1)$ possible linearly independent error eigenfunctions; however, the replacement of (r, s) by $(p - r, q - s)$ merely changes the sign of t, hence multiplies the two A-roots by -1. Thus (24), (26), and (27) define just the $(p-1)(q-1)$ linearly independent eigenfunctions required to form a complete set.

If we again take $\alpha = \frac{1}{4}$ the roots of (27) are

$$A = 0, \frac{t}{2}.$$

The value $A = 0$ corresponds to the complete removal of an isolated error occurring at $j = 1$ or $k = 1$. The other root gives

$$K = A^2 = \tfrac{1}{4}t^2 = \tfrac{1}{4}[\cos(\pi r/p) + \cos(\pi s/q)]^2. \tag{28}$$

The greatest eigenvalue occurs both for $(r = s = 1)$ and for $(r = p - 1, s = q - 1)$, and is,

$$K^* = \tfrac{1}{4}[\cos(\pi/p) + \cos(\pi/q)]^2 \cong [1 - \pi^2(p^{-2} + q^{-2})/4]^2.$$

Thus in each iteration cycle the most resistant errors are reduced as much as in two cycles in the Richardson procedure. (This doubling of efficiency is qualitatively plausible since the numbers from which each $\phi_{j,k}^{n+1}$ is compounded are in the mean half as "old" in the Liebmann as in the Richardson procedure.)

5. The Extrapolated Liebmann Method. Since the eigenvalues, K, are all positive it seems plausible that increasing α should improve the efficiency of this process. The procedure so obtained we may term the "extrapolated Liebmann method." Examination of equation (26) shows that an improvement in convergence rate can be so obtained. For $\alpha > \frac{1}{4}$ the constant term is positive. We may therefore distinguish two ranges of α, hence of the middle term. Where

$$\alpha^2 t^2 \leqslant 4\alpha - 1; \quad t^2 \leqslant (4\alpha - 1)/\alpha^2 \tag{30}$$

the two roots of (26) are complex conjugates having the magnitude

$$|A| = (4\alpha - 1)^{\frac{1}{2}}; \quad |K| = 4\alpha - 1, \tag{31}$$

which increases with increasing α. For

$$(4\alpha - 1)/\alpha^2 < t^2 < 4; \quad \alpha < \tfrac{1}{2} \tag{32}$$

the two roots for A are real, one of greater magnitude than $(4\alpha - 1)^{\frac{1}{2}}$ (but <1), the other less. Differentiating (27) gives

$$\frac{dA}{d\alpha} = \frac{At - 2}{A - t\alpha}; \quad At - 2 < 0; \quad A - t\alpha = \pm [\alpha^2 t^2 - (4\alpha - 1)]^{\frac{1}{2}}, \tag{33}$$

thus in the range (32) the greater of the magnitudes of the roots decreases with increasing α. The optimum value of α is therefore that for which the range (30) of complex roots just covers the spectrum of t-values. For the largest and smallest t-values

$$t = \pm [\cos(\pi/p) + \cos(\pi/q)] = \pm t_{\max}, \tag{34}$$

thus for α we use the smaller root of

$$\alpha^2 t_{\max}^2 - 4\alpha + 1 = 0. \tag{35}$$

Then for all of the error eigenfunctions

$$|K| = K^* = 4\alpha - 1. \tag{36}$$

For large p and q this may be approximated by

$$K^* \cong 1 - \sqrt{2}\pi(p^{-2} + q^{-2})^{\frac{1}{2}}. \tag{37}$$

With this procedure and with the optimum values of α the number of iterations required to produce a substantial improvement in a trial solution increases about linearly with p and q rather than quadratically as it does for the Richardson method or the Liebmann method using $\alpha = \frac{1}{4}$.

The usefulness of the extrapolated Liebmann method is limited by the difficulty of determining the optimum α-value for more complex problems than the example considered here. It seems likely, however, that in many similar problems a considerable improvement in convergence rate can be achieved by a suitable choice of α and that an approximate optimum α-value can be found empirically without great difficulty.

The Liebmann and extrapolated Liebmann methods have an advantage —for some types of machine applications—over the Richardson and similar methods in that they require carrying as machine "memory" no more than one complete set of $\phi_{j,k}$-values. However, for use with punched card machines or other calculating machines with severely limited internal (rapid-access) memory this advantage is offset by the difficulty of retaining the newly calculated values for use in the succeeding point and (more particularly) for the adjacent point in the succeeding row (or column). For such machines it is more convenient to use procedures in which a ϕ-value when calculated may be stored in the "external memory" until needed in the next iteration cycle.

6. The Second-Order Richardson Method. An improvement in the convergence rate of the Richardson method comparable to that achieved for the Liebmann method by extrapolation may be gained by retaining for use in the calculation of $\phi_{j,k}^{n+1}$ not only $\phi_{j',k'}^{n}$ (j', k' running over points neighboring j, k) but also $\phi_{j,k}^{n-1}$. This modification is suggested by the formal equivalence between the Richardson method and the solution of the time-dependent equation, $\dfrac{\partial\phi}{\partial t} = L\phi$. The modified iteration method is equivalent to the solution of an equation of second order in t,

$$\frac{\partial^2\phi}{\partial t^2} + A\frac{\partial\phi}{\partial t} - L\phi = 0,$$

and may therefore be termed the "second-order Richardson method." The iteration process may then be written as

$$(38)\qquad \begin{aligned} \phi_{j,k}^{n+1} &= \phi_{j,k}^{n} + \alpha L\phi_{j,k}^{n} + \beta(\phi_{j,k}^{n} - \phi_{j,k}^{n-1}) &&\text{(interior points)}\\ &= b_{j,k} &&\text{(boundary points).}\end{aligned}$$

We again denoted by $\phi_{j,k}$ the solution of the equation $L\phi_{j,k} = 0$ satisfying the boundary condition incorporated in (38) and by $\epsilon_{j,k}^{n}$ the difference, $\phi_{j,k}^{n} - \phi_{j,k}$. The error then satisfies the induction equation (38) with a zero boundary condition. An error-eigenfunction, $\epsilon_{j,k}^{\nu}$, of L belonging to the eigenvalue, $L_{(\nu)}$,

$$(39)\qquad \begin{aligned} L\epsilon_{j,k}^{\nu} &= L_{\nu}\epsilon_{j,k}^{\nu} &&\text{(interior points)}\\ \epsilon_{j,k}^{\nu} &= 0 &&\text{(boundary points),}\end{aligned}$$

is then multiplied in each iteration cycle by a factor, K, such that,

$$K_{\nu}^{n+1}\epsilon^{(\nu)} = K_{\nu}^{n}\epsilon^{(\nu)} + \alpha L_{\nu}K_{\nu}^{n}\epsilon^{(\nu)} + \beta(K_{\nu}^{n} - K_{\nu}^{n-1})\epsilon^{(\nu)}$$

or (dropping the subscript, ν)

$$(40)\qquad K^2 - (1 + \alpha L + \beta)K + \beta = 0.$$

If all of the eigenvalues, L_ν, are of the same sign (which for definiteness we take as negative) and are in the range

(41) $$0 > L_0 \geqslant L \geqslant L_m,$$

then α and β can be chosen to minimize

(42) $$K^* = \max|K_\nu| < 1.$$

(Since we are here using a second-order recurrence relation for ϕ^n the two roots of (40) correspond to two separate modes of decay for each eigenfunction. Both decay rates must be considered in (42).) For positive β we again have a range of values of αL within which the roots of (40) are complex and of magnitude $\beta^{\frac{1}{2}}$. It may be shown, as before, that outside this range the greater $|K|$ decreases with increasing β. Thus α and β should be chosen to make the range of complex roots just fit the range (41). This requires

$$1 + \alpha L_0 + \beta = 2\beta^{\frac{1}{2}}$$
$$1 + \alpha L_m + \beta = -2\beta^{\frac{1}{2}},$$

hence

$$\alpha(L_0 + L_m) = -2(1 + \beta)$$
$$\alpha L_0 = -(1 - \beta^{\frac{1}{2}})^2,$$

or

(43) $$\eta = L_0/(L_0 + L_m) = \tfrac{1}{2}(1 - \beta^{\frac{1}{2}})^2/(1 + \beta); \quad \beta - 2\beta^{\frac{1}{2}}(1 - 2\eta)^{-1} + 1 = 0.$$

Choosing the smaller root gives for the decay factor,

(44) $$K^* = \beta^{\frac{1}{2}} = 1 - 2(1 - 2\eta)^{-1}(\eta - \eta^2)^{\frac{1}{2}} \cong 1 - 2\eta^{\frac{1}{2}}.$$

For the rectangular Laplace problem

$$\eta \cong \pi^2(p^{-2} + q^{-2})/8,$$

hence

(45) $$K^* = 1 - \pi(p^{-2} + q^{-2})/\sqrt{2}.$$

Thus a comparison of the convergence rate of the second-order Richardson method (45) with the extrapolated Liebmann method (37) again shows a factor of two advantage over the latter.

It is not in general possible, to the writer's present knowledge, to effect a comparable further improvement in the Richardson method by using a third or higher order induction process.

7. Applications to Eigenvalue Problems. The methods here described may be used to determine the fundamental eigenfunction of an equation of the form

$$(L + \lambda^2)\phi = 0$$

with homogeneous boundary conditions if the smallest value of λ^2 is approximately known. The operator, L, is replaced throughout by $L + \lambda^2_{0,\text{approx.}}$ and an initial trial solution chosen to approximate the fundamental eigenfunction. Then the higher eigenfunction components of the trial solution will decay during the iteration process while the fundamental remains approximately unchanged (depending on the accuracy of $\lambda_{0,\text{approx.}}$). Similarly a higher

eigenfunction can be calculated by replacing L by $(L + \lambda^2_{j,\text{approx.}})^2$. It is to be expected that the iteration method so obtained would be very slowly convergent since η is the square of the ratio of the largest to the smallest value of $|L + \lambda^2_{i,\text{approx.}}|\,(i \neq 0)$.

8. Time Estimates. In order to indicate the order of difficulty of typical problems to which these methods are applicable we consider two problems; the solution of the Laplace equation and of the biharmonic equation in a square region,

$$p = q = N.$$

The possibility, for some forms of the boundary conditions, of treating the biharmonic equation by factoring into a Laplace and a Poisson equation, will not be considered.

We assume, for definiteness, that the iteration process is to be continued until all errors are reduced by a factor of 10^{-6}. The number of cycles of iteration required will then be approximately,

$$\eta_{\max} \cong 6/|\log K^*|$$

while the number of ϕ-values to be corrected in each iteration is $(N - 1)^2$.

We denote by τ the mean time required for each arithmetic operation. Then the time required for the evaluation of one $\phi^n_{j,k}$ is T where

Method	T
Laplace-Richardson	4τ
Laplace-Liebmann ($\alpha = \frac{1}{4}$)	4τ
Laplace-Liebmann (optimum α)	7τ
Laplace-2nd order Richardson	9τ
Biharmonic-Richardson	12τ
Biharmonic-Liebmann (with extrapolation)	15τ
Biharmonic-Liebmann (no extrapolation)	12τ
Second-Order Richardson	17τ

The total calculating time is given in the following table for a 10×10 and for a 20×20 lattice, and the asymptotic form for large N

Approximate Total Calculating Time

	$N = 10$	$N = 20$	N Large
Laplace-Richardson	$9\cdot10^4\tau$	$1.6\cdot10^6\tau$	$11\ N^4\tau$
Laplace-Liebmann ($\alpha = \frac{1}{4}$)	$5\cdot10^4\tau$	$8\cdot10^5\tau$	$5.6\ N^4\tau$
Laplace-Liebmann (optimum α)	$1.2\cdot10^4\tau$	$1.1\cdot10^5\tau$	$15\ N^3\tau$
Laplace-Second-Order Richardson	$3\cdot10^4\tau$	$3\cdot10^5\tau$	$40\ N^3\tau$
Biharmonic-Richardson	$1.1\cdot10^7\tau$	$8\cdot10^8\tau$	$14\ N^6\tau$
Biharmonic-Second-Order Richardson	$4\cdot10^5\tau$	$7\cdot10^6\tau$	$48\ N^4\tau$
Biharmonic-Liebmann	?	?	?

For the biharmonic equation the ratio, $\eta = L_0/(L_0 + L_m)$, has been approximated by the square of the corresponding value for the Laplace equation.

To convert these values to true time estimates we may, quite crudely, approximate τ by 10^{-5} days for electromechanical computers (e.g., punched card machines) and by 10^{-8} days for entirely electronic computers. It is thus seen that with a fairly fine mesh the calculating time required with the slower machines is uncomfortably large for the Laplace equation and prohibitive for the biharmonic equation if the normal Richardson method is

used. Even with the faster machines the time required for the solution of a biharmonic equation by the methods considered here is uncomfortably large unless the second order Richardson (or probably also the extrapolated Liebmann) method is used. It is clear that for many problems of interest the simplest iterative procedures will prove impossibly tedious even with the fastest automatic computers.

The apparent likelihood that the extrapolated Liebmann procedure would prove more rapidly convergent and more convenient for electronic computers than the second-order Richardson method would seem to justify an experimental study with such a computer.

The writer is indebted to R. H. MacNeal and W. E. Milne for encouragement and helpful discussion.

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[2] R. V. Southwell, *Relaxation Methods in Engineering Science*, Oxford, 1940.

[3] E. T. Whittaker & G. Robinson, *The Calculus of Observations*, London and Glasgow, 1932.

[4] G. Shortley, R. Weller, & B. Fried, "Numerical solution of Laplace's and Poisson's equations with applications to photoelasticity and torsion," Ohio State University, *Studies, Engineering Series, Bull.* no. 107, 1942.

[5] L. F. Richardson, "The approximate arithmetical solution by finite differences of physical problems involving differential equations, with an application to the stresses in a masonry dam," R. Soc., London, *Phil. Trans.* s. A, v. 210, 1911, p. 307–357. "How to solve differential equations approximately by arithmetic," *Math. Gazette*, v. 12, p. 415–421, 1925.

[6] R. Courant, "Uber partielle Differenzengleichungen," Congresso Internazionale dei Matematici, *Atti*, Bologna, v. 3, 1930, p. 83–89.

[7] This can be shown by the method used by H. Lewy in "On the convergence of solutions of difference equations," *Studies and Essays Presented to R. Courant on his 60th Birthday*, New York, 1948, p. 211–214.

[8] H. Liebmann, "Die angenäherte Ermittelung harmonischer Funktionen und konformer Abbildungen," Bayer. Akad. Wiss., *math.-phys. Klasse, Sitz.*, 1918, p. 385–416 Further references to the Liebmann method are cited in footnote 1 of Shortley, Weller & Fried. See also *MTAC* v. 3, p. 350, footnote 3.

On a Definite Integral

The function

$$f(x) = \int_0^\infty (u + x)^{-1} \exp(-u^2) du$$

is tabulated by E. T. Goodwin and J. Staton[1] [*MTAC*, v. 3, p. 483] from series expansions and by numerical integration of the differential equation satisfied by the function.

The integral can be evaluated explicitly in terms of two simple tabulated functions. The writer used the Laplace transform[2] to evaluate the integral. This method will now be given.

We may write

$$(1) \quad I(t) = \int_0^\infty (u + t^{\frac{1}{2}})^{-1} \exp(-u^2) du = \int_0^\infty (1 + v)^{-1} \exp(-tv^2) dv$$

and if we let

$$L\{I(t)\} \equiv \int_0^\infty I(t) \exp(-pt)dt$$

denote the Laplace transform of $I(t)$, we have

$$(2) \quad L\{I(t)\} = \int_0^\infty (p + v^2)^{-1}(1 + v)^{-1}dv = \tfrac{1}{2}(1 + p)^{-1}[\pi p^{-\frac{1}{2}} + \ln p] = \phi(p)$$

so that by the inverse transformation

$$(3) \quad I(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \phi(p) \exp(pt)dp$$

$$(4) \quad = \frac{e^{-t}}{4\pi i}\int_{c-i\infty}^{c+i\infty} z^{-1}e^{zt} \ln(z - 1)dz + \frac{1}{4i}\int_{c-i\infty}^{c+i\infty} z^{-\frac{1}{2}}e^{zt}(1 + z)^{-1}dz.$$

The value of these integrals may be found by referring to tables of transforms.[3,4] The result is

$$(5) \quad I(t) = \tfrac{1}{2}e^{-t}\int_{-\infty}^{t} u^{-1}e^{u}du + \pi^{\frac{1}{2}}e^{-t}\int_0^{t^{\frac{1}{2}}} \exp(u^2)du$$

so that*

$$(6) \quad f(x) = \tfrac{1}{2}\exp(-x^2)Ei(x^2) + \pi^{\frac{1}{2}}F(x),$$

where

$$Ei(y) = \int_{-\infty}^{y} u^{-1}e^{u}du \quad \text{and} \quad F(y) = \exp(-y^2)\int_0^y \exp(u^2)du$$

are well-tabulated functions.[5] By expanding this result in an ascending series it can be identified term by term with equation (6) of Goodwin and Staton's paper.

I have communicated this result to Dr. Goodwin who informed me that although not aware of it he was within one step of deriving it from equation (4) of his paper. The following derivation is due to Dr. Goodwin:

Equation (4) gives

$$f'(x) + 2xf(x) = \pi^{\frac{1}{2}} - x^{-1}.$$

Thus

$$(7) \quad d[\exp(x^2)f(x)]/dx = \pi^{\frac{1}{2}}\exp(x^2) - x^{-1}\exp(x^2).$$

If this could be integrated between suitable limits it would give f in terms of $Ei(x^2)$ and $F(x)$ but, in fact, it is impossible to choose such real limits and it was at this point that Dr. Goodwin stopped. However, equation (7) can be written

$$(8) \quad \frac{d}{dx}\left\{\exp(x^2)f(x) - \pi^{\frac{1}{2}}\int_0^x \exp(u^2)du\right\} = -x^{-1}\exp(x^2).$$

Taking the indefinite integral

$$(9) \quad \exp(x^2)f(x) - \pi^{\frac{1}{2}}\int_0^x \exp(u^2)du = -\int^x u^{-1}\exp(u^2)du + c$$

$$(10) \quad = -\frac{1}{2}\int^{x^2} t^{-1}e^{t}dt + c.$$

Comparison of the known behavior of $f(x)$ as $x \to 0$, namely

$$f(x) \sim -\ln x - \tfrac{1}{2}\gamma$$

with the fact that $Ei(x^2) \sim \ln x^2 + \gamma$ shows that the constant in equation (10) can be taken as zero if the lower limit of the integral is taken as $-\infty$. Equation (6) follows immediately.

This corresponds to the use of a limit of $i\infty$ when (7) is integrated so that it is not surprising that Dr. Goodwin missed the result.

I should like to express my thanks to Dr. Goodwin for permission to use his method in the latter part of this paper.

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[1] E. T. GOODWIN & J. STATON, "Table of $\int_0^\infty e^{-u^2}du/(u+x)$," *Quart. Jn. Mech. Appl. Math.*, v. 1, 1948, p. 319–326.
[2] H. T. CARSLAW & J. C. Jaeger, *Operational Methods in Applied Mathematics*, Oxford, 1941.
[3] L. A. PIPES, *Applied Mathematics for Engineers and Physicists*, McGraw-Hill, New York, 1946.
[4] R. V. CHURCHILL, *Modern Operational Mathematics in Engineering*, McGraw-Hill, New York, 1944.
[5] FMR *Index*, p. 189–191, 220, 231.
[*] An ambiguity in this result will be discussed in the next issue—ED.

RECENT MATHEMATICAL TABLES

723[A–Z].—HAROLD T. DAVIS & VERA FISHER, *A Bibliography of Mathematical Tables. Copyright, 1949, by Harold T. Davis*, xxii, 286, 27 leaves. For sale University Bookstore, Northwestern Univ., Evanston, Ill., bound \$4.00. Mimeographed one side of each sheet. 20.9 × 27.5 cm.

This copyright work has been issued in an edition of only 50 copies in order to test the possible demand for a revised and more complete volume of this kind. The title page states that it was "prepared under the direction of" Professor Davis "with the assistance of" Miss Fisher. We shall try, in a general way, to articulate clearly what is to be found in this tentative edition and to make some (among many possible) comments, which may serve as useful suggestions in any later revisions.

The volume is divided into four parts:

I. Introduction, leaves i–xxii.
II. Bibliography of Mathematical Tables, leaves 1–196 + 27.
III. Index of Tables by Classification of Functions, leaves 197–262.
IV. Index of Tables by Names of Authors, leaves 263–286.

We shall begin by first considering the 223 pages of Section II. There are here about 3,680 entries dated 1475–1948. Titles of almost all periodical articles are given, which is a highly praiseworthy feature, but there is a most undesirable total lack of uniformity of treatment of such titles, as well as of the abbreviated forms for titles of periodicals, and other details. In the case of pamphlets or books there is a similar lack; sometimes the total number of pages in the volume is given but more often not; names of places of publications are omitted in a number of entries, and generally unintelligible Latin forms are also to be found; there is no uniformity in the forms of names of authors (at least six of which are incorrectly spelled), and titles such as

"Major General" and "Sir" are given in some cases but wholly lacking in dozens of others; dates of publication are absent in a number of cases; in at least one case the library and location of a publication is given, and in a few cases some inkling is noted as to the contents of entries, but the exact pages in books, where material of interest in the bibliography appears, are practically never given; the date of a third edition of an important work is given as if it were the first, and similarly for a third tirage; there are many titles with errors, some of the most fantastic character. There are at least three cases where two works of each of three authors are each ascribed to two different people. A number of dates of publications are incorrect. Wrong tabular references to certain authors are to be found in at least two cases.

The arrangement of titles is alphabetical according to authors, except in such cases as reports of the Table Committee of the BAAS (where the authors of tables are almost completely known), Harvard University, Computation Laboratory, NBSCL, etc. For each of 20 letters of the alphabet, one to three extra pages (27 in all), numbered xa, xb, xc, with additional titles, are added.

The list gives evidence that the method of compilation included the copying (not always accurately) of practically every entry of published material in:

(a) FMR, *Index*, 1946 (with the addition of titles in the case of periodical references);
(b) D. H. Lehmer, *Guide to Tables in the Theory of Numbers*, 1941;
(c) *Scripta Mathematica*, v. 2 (1934), p. 91–93, 297–299, 379–380; v. 3 (1938), p. 97–98, 192–193, 282–283, 364–366; v. 4 (1936), p. 101–104, 198–201, 294–295, 338–340.
(d) J. Henderson, *Bibliotheca Tabularum Mathematicarum*, 1926;
(e) J. W. L. Glaisher, *Report of the Committee on Mathematical Tables*, 1873; and some titles from:
(f) R. C. Archibald, *Mathematical Table Makers*, 1948;
(g) *MTAC*, nos. 1–26, 1943–1949;
(h) A. De Morgan, "Table," *The English Cyclopaedia, Arts and Science Sect.*, v. 7, 1861.

These sources certainly account for more than 90% of the titles. The reviewer has verified that of the 3,680 entries there are 1,085 in the letters A–F and that of these 94% of the entries were in (a)–(h). In the 6% of new titles 36 referred to tables, 21 to theory, and 6 to calculating machines (including slide rules). In the whole book there are references to over 130 authors of material on calculating machines, and 16 references to graphical aids. It will thus be seen that the title of Section II is a decided misnomer and that the title of the work under review is only partially descriptive.

Because of the nature of the compilation there are scores of titles referring to material of no possible current use and almost wholly inaccessible to nearly every reader. There is no such exclusion as was practiced in the FMR, *Index*. In its present form Section II is wholly unreliable as a historical or bibliographical guide. An enormous amount of labor will still be necessary to bring this fundamental Section into a form having appeal to scholars.

With Section II we next naturally associate the 24 pages of Section IV and the 12 pages of Subject Classification, p. xi–xxii, of I. In IV is an alpha-

betical list of authors followed by indications of the kind of tables, or other contents, in the entries of II, according to the useful scheme of classification, which is a great elaboration of the classification given by Professor Davis in his *Tables of the Higher Mathematical Functions*, v. 1, 1933, p. 5–12. In a general way the new classification was that adopted by the NRC Committee on Mathematical Tables and Other Aids to Computation.

In Section III (64 p.), under about 300 subject headings are references by authors to the entries in IV.

In practically no case is there any indication of the range of any table. No warning is anywhere given concerning wholly useless tables, because of their gross inaccuracies.

If, in the whole work, 6% of the titles are new items for consideration by the authors of the possible forthcoming new edition of FMR, *Index*, a valuable contribution to scholarship will have been made. Furthermore, the ready subject index of Section III may, with the aid of other works, be frequently useful to specialists. Misspelled "Napierian" on p. xi and 202 should be wholly eliminated.

In the Introduction is a table showing the "distribution by centuries of 3,410 contributions to tables and table-making," for the period 1500–1947. The difference between 3,410 and 3,680 appears to indicate the number of titles published in 1948. For the century 1500–1599 two titles are indicated, presumably for Rheticus and Otho (listed as a separate title and not as part of the work of Rheticus); Regiomontanus, 1475, is not counted. The numbers are indicated for the following quinquennial periods; for the century 1600–1700 the total number of titles is given as 70; 1701–1800, 84; 1801–1900, 985; 1901–1947, 2,262.

It has not seemed worth-while to give chapter and verse for every statement made in this review. In a work of this kind a "prepared-under-the-direction-of" prop is indeed a frail one whereon to lean.

R. C. A.

724[A, B, D, E, G, H, I].—FRITZ EMDE, *Tables of Elementary Functions. Second Edition. With 83 Figures. Tafeln elementarer Funktionen.*, Leipzig, Teubner, 1948, xii, 181 p. 16.3 × 24 cm. Price 11.60 German marks, bound in boards and canvas.

This work, first published in 1940, was a great elaboration of the 78 pages devoted to Elementary Functions in the third edition (1938) of JAHNKE & EMDE's *Tables of Functions*. Professor Emde has the following preface in the present edition:

"The Second Edition of the Tables of Elementary Functions (an almost unchanged reprint of the First Edition of 1940) should have been issued in 1944. But after having been printed all copies were destroyed at the book binder's by bombs and fire during the war. It is only now possible to reprint this edition from the same manuscript. Pretzfeld, January 1948"

An offset reprint of the 1940 edition was made at Ann Arbor in 1945, and this was reviewed in *MTAC*, v. 1, p. 384–385.

The only difference which the reviewer noticed, in comparing this edition with the work under review, was the substitution, on p. 10, of a fine relief (Fig. 3) of the function z^z, z being complex (instead of $z = x^y$ on logarithmic

scale for $-1 < y < +1$) supplementing the altitude chart, Fig. 79, p. 158 [the function $z^s = (re^{i\varphi})^{s+i\varphi}$].

R. C. A.

725[B, L].—H. BREMMER, *Terrestrial Radio Waves. Theory of Propagation.* New York, Elsevier Publishing Co., 1949, x + 343 p. 17 × 24.5 cm. Price $5.50.

Page 44 contains 3D values of $\frac{1}{2}(3x)^{\frac{2}{3}}$, where x is either $S + \frac{1}{4}$ or $S + \frac{3}{4}$ and $S = 0(1)5$. Page 45 contains 3D values of $\frac{1}{2}(3x)^{\frac{2}{3}}$, where the x are the first six positive roots of $J_{2/3}(x) - J_{-2/3}(x) = 0$ or $J_{1/3}(x) + J_{-1/3}(x) = 0$. Page 69 is a relief diagram of $H_{1/3}(x + iy)$, for $-10 \leqslant x \leqslant 10$ and $-5 \leqslant y \leqslant 5$. The branch cut is on the negative real axis.

Chapter VI. "Numerical computations and results." The results are mostly given in the shape of careful diagrams of which there is a large number. Other similar diagrams can be found throughout the book.

A. E.

726[C, L].—HARVARD UNIVERSITY, COMPUTATION LABORATORY, *Annals*, v. 22. *Tables of the Function* $\dfrac{\sin\phi}{\phi}$ *and of its First Eleven Derivatives.* Cambridge, Mass., Harvard University Press, 1949, xviii, 241 p. 19.6 × 26.6 cm. Price $8.00.

This interesting volume was prepared as an aid to the application of FOURIER transforms. It gives to 9D the functions

$$g(\phi) = \phi^{-1}\sin\phi$$

and $g^{(n)}(\phi)$, for $n = 1(1)11$ and $\phi = 0(\pi/360)20\pi - \pi/360$.

The argument ϕ is expressed in degrees, the interval being $\frac{1}{2}°$. The table is conveniently arranged so that at one opening there are 14 columns, the extreme ones giving the argument and the 12 others giving $g^{(n)}(\phi)$ for $n = 0(1)11$. The functions are very moderate in their behavior. In fact the bulk of the table gives only 8 significant figures. Beyond p. 38 all values are less than a tenth. However, the functions are still very much alive at 20π.

Since the tables give successive derivatives, interpolation is most easily performed by TAYLOR's formula, the second-order formula being adequate for 8D work, linear interpolation giving accuracy to within 4 units of the 6-th decimal.

The table was computed on Harvard's Mark I from an 18D table of $\sin\phi$, the successive g's were computed from the recurrence formula

$$g^{(n)}(\phi) = \left[\frac{d^n\sin\phi}{d\phi^n} - ng^{(n-1)}(\phi)\right]\Big/\phi.$$

The whole computation required less than two weeks of machine time.

An introduction (p. xiv–xviii) by R. C. SPENCER includes a few of the applications of the table to the theory of Fourier transforms, although no actual examples are worked out. Briefly, there are two main uses which facilitate the transformation of a function defined or given by a power series (really a polynomial of degree $\leqslant 12$) or a Fourier series (really a trigonometric polynomial of degree $\leqslant 20$). These methods apply to functions

$F(x)$ of the real variable x, which vanish outside a finite interval, say $-1 < x < 1$. Then the Fourier transform G of F is given by

$$G(u) = \int_{-1}^{1} F(x)e^{iux}dx.$$

In case

$$F(x) = \sum_{n=0} A_n x^n,$$

then

$$G(u) = 2 \sum_{n=0} A_n i^{-n} g^{(n)}(u),$$

and in case

$$F(x) = \sum_{n=0} A_n e^{\pi i n x},$$

then

$$G(u) = 2 \sum_{n=0} A_n g(u + n\pi).$$

Thus, the table is used in the first case by rows and in the second case by its first column to obtain directly the values of the transformed functions.

D. H. L.

727[D, P].—Jules Gaunin, L. Houdaille, & A. Bernard, *Tables pour le Tracé des Courbes de Chemins de Fer, Routes & Canaux. Nouvelle édition revue et corrigée. Nouveau tirage. Première Partie: Tables Trigonométriques . . . Deuxième Partie: Recueil de Coordonnées* . . . Paris, Dunod, 1948, xlvii, 181 p., xiv, 182 p. 13.7 × 21 cm.

This is a new tirage of a very old book; it seems to be practically identical with the edition of 1922, when the chief author Gaunin was already dead; there was another tirage in 1925. According to the *Catalogue* of the Bibliothèque Nationale there were a 1919 edition containing 426 p., a two-v. edition in 1911; and the second ed. in 1904, 2 parts in one v. The first edition of *Tables Trigonométriques pour le tracé des Chemins de Fer* . . . , was published in Paris, Dunod, 1862, xxxii, 181 p.; the second part, *Recueil de Coordonnées*, by Jules Gaunin, L. Houdaille, & A. Bernard, Paris, 1896, xxvi, 176 p.

The first part gives a 6D table of the six trigonometric functions, versed sine, and versed cosine, for $\alpha = 0(30'')90°$. There are also several other columns with values of 2α, $180° - 2\alpha$, $90° - \alpha$, 6D values of $\pi\alpha/180°$ and of $\pi(90° - \alpha)/180°$. The introductory pages deal with material of the tables, with trigonometry, and with some practical problems.

The second part is devoted to tables for determining coordinates of points on circular arcs. The tables on p. 1–120 are for finding ordinates corresponding to abscissae measured on a tangent to the arc to be determined. The tables on p. 121–154 are for solving similar problems with reference to a chord of the arc; and the tables, p. 155–156, for points corresponding to a prolongation of the chord.

Of the five remaining tables, A–E, in A, for $R = 100(5)600(100)$-$3000(500)4000$, 9D values are given for $2\pi R$; lengths of arcs corresponding to central angles, 1°, 1′, 1″, i.e., $2\pi R/360$, $2\pi R/21600$, $2\pi R/1296000$; and angles in seconds corresponding to an arc of 1^m; also the values of $1000/R$.

In B, for $R = 1^m$, corresponding to 1(1)100, interpreted as central degrees, or minutes, or seconds, are given the corresponding lengths of arcs in meters.

In C, for arc of .01(.01)1(1)10(10)100(100)1000 meters, and $R = 100(50)$-600(100)1000, 1.1(.1)3(.5)4(1)6, 10, are given the corresponding angles to the nearest thousandth of a second.

Tables D and E are for conversion of degrees to grades, and for grades to degrees.

We have elsewhere referred to such easement curves as the clothoid which are the ones of importance in modern times in laying out railways and other routes; see *MTAC*, v. 3, p. 146, 452.

R. C. A.

728[F].—H. CHATLAND, "On the euclidean algorithm in quadratic number fields," Amer. Math. Soc., *Bull.*, v. 55, 1949, p. 548–553.

For each prime p of the form $24x + 1$ less than $2^{14} = 16384$ with the exception of $p = 73, 97, 193, 241, 313, 337, 457$ and 601 there is given a representation

(1) $$p = q_1m_1 + q_2m_2,$$

where q_1, q_2, m_1, m_2 are quadratic nonresidues of p, and where q_1 and q_2 are odd primes dividing q_1m_1 and q_2m_2 respectively to odd powers. No such representations are possible in the exceptional cases of p listed above.

No column headings indicate which numbers in the table are the q's. These seem to be the first and third numbers on the right of the equal sign, except in the case of $p = 409$ and 577.

ERDÖS & KO[1] showed that in case the prime p has a representation (1) the field $K(p^{\frac{1}{2}})$ has no euclidean algorithm. The reason for the upper limit 2^{14} is that according to a theorem of DAVENPORT,[2] no field $K(p^{\frac{1}{2}})$ is euclidean for $p > 2^{14}$.

D. H. L.

[1] P. ERDÖS & CH. KO, "Note on the euclidean algorithm," London Math. Soc., *Jn.* v. 13, 1938, p. 3–8.

[2] H. DAVENPORT, "Indefinite binary quadratic forms and Euclid's algorithm in real quadratic fields," to appear in London Math. Soc., *Proc.*

729[F].—S. D. CHOWLA & J. TODD, "The density of reducible integers," *Canadian Jn. Math.*, v. 1, 1949, 297–299.

A definition of a reducible integer is given in RMT **734**. An alternative definition is the following. An integer is reducible in case the prime factors of $1 + n^2$ are all less than $2n$. The authors in collaboration with J. W. WRENCH have examined the first 5000 numbers and find that about 30 per cent of them are reducible. This conjecture is unproved. The number of reducible integers in each of the 50 centuries is tabulated. The table also gives the number B_n of integers n whose prime factors are less than $2n^{\frac{1}{2}}$. These numbers are shown to have a density

$$1 - \ln 2 = .3068528.$$

The table has been completely recalculated by one of the authors and was found to contain 25 errata. Since the table contains only 12 lines it is more economical to reproduce it below in corrected form than to single out its

errata. The number A_n of reducible integers $<n$ is given on the left, not on the right as stated in the paper.

	0		1000		2000		3000		4000			
n	A_n	B_n	A_n	B_n	A_n	B_n	A_n	B_n	A_n	B_n		
1–100	30	57	31	43	29	43	35	41	28	42		
101–200	29	50	27	43	30	42	28	43	28	40		
201–300	27	47	33	44	23	42	24	43	28	41		
301–400	26	45	27	41	32	39	32	43	31	40		
401–500	31	45	31	45	27	44	28	41	27	42		
501–600	29	45	23	44	32	39	34	41	37	39		
601–700	29	44	27	40	26	43	24	40	33	41		
701–800	29	44	35	43	32	41	30	43	35	39		
801–900	27	44	27	45	27	43	29	40	30	43		
901–1000	23	42	31	39	29	42	20	41	38	41		
Totals	280	463	292	427	287	418	284	416	315	408	1458	2132

D. H. L.

730[F].—L. Gambelli, "Sui caratteri di divisibilita con una tabella dei coefficienti di divisibilita di tutti i numeri da 2 a 101," *Period. Mat.*, s. 4, v. 27, 1949, p. 109–116.

This note gives a table of the absolutely least remainder of 10^n on division by m for all values of n and each integer $m \leqslant 101$. In case this remainder is negative it is printed in boldface type. The purpose of the table is to give rules for finding the remainder on division of a given number N by m by replacing N by a linear combination of its decimal digits, the coefficients of this combination being tabulated. These coefficients are periodic, preceded by a preliminary nonperiodic part whenever m is not prime to 10.

As an example of the use of the table, for $m = 37$ the table gives the period

$$1, 10, -11, 1, 10, -11, \cdots.$$

The corresponding criterion for divisibility by 37 is as follows: Let the digits of N, beginning from the left, be $d_0, d_1, d_2, \cdots$. Then N is divisible by 37 if and only if

$$d_0 + 10d_1 - 11d_2 + d_3 + 10d_4 - 11d_5 + \cdots$$

is divisible by 37.

D. H. L.

731[F].—H. Gupta, "On a conjecture of Miller," Indian Math. Soc., *Jn.*, v. 13, 1949, p. 85–90.

The well-known function $\mu(n)$ of Möbius, which is $+1$ or -1 according as n is a product of an even or odd number of distinct primes and which is zero otherwise, plays an important role in the theory of distribution of primes. Stieltjes conjectured in 1885 that the function

$$M_1(x) = \sum_{n \leq x} \mu(n)$$

satisfies

$$|M_1(x)| \leqslant x^{\frac{1}{2}},$$

whose truth would imply the truth of the celebrated Riemann hypothesis. The conjecture mentioned in the title of the present paper involves the sum

function

$$M_2(x) = \sum_{n \le x} M_1(n)$$

and asserts that $M_2(x) \leqslant 0$ for $x \geqslant 3$. The paper verifies this conjecture for $x \leqslant 20\,000$. A table is given of $-M(x)$ for $x = 25(25)20\,000$. There is also a table of $-A(x) = -M(x)/x$ for $x = [100(100)20\,000; 5D]$.

The author conjectures that $A(x)/\log x$ is bounded. According to a heuristic argument of BRUN,[1]

$$A(x) = -2 + 12/x + \cdots.$$

The present table fails to support this result since $A(x) < -4$ for x near 18500.

D. H. L.

[1] VIGGO BRUN, "La somme des facteurs de Möbius," Den 10. Skandinaviske Matematiker Kongres, *Comptes Rendus*, Copenhagen, 1947, p. 40–53.

732[F].—M. PETROVICH, "Elementarna posmatrania o rasporedu omaniih̑k prostik̑h brojeva," [Elementary observations on the distribution of small prime numbers], Srpske Akad. Nauka, Belgrade, *Glas*, v. 189, 1946, p. 5–45.

Page 43 contains tables of the number of primes not exceeding x and the number of these which are of the forms $6m + 1$ and $6m - 1$ for $x = 50(50)1000$. These functions are compared with certain approximating functions.

733[F].—H. TIETZE, "Tafel der Primzahl-Zwillinge unter 300 000," Akad. d. Wissen., Munich, *math. nat.-Kl., Sitz.*, 1947, p. 57–72 (1949).

If p and $p + 2$ are both primes, then $(p, p + 2)$ is called a prime pair or a pair of twin primes. The table in this paper lists the prime pairs $<300\,000$ by giving $p + 2$ in each case. This makes 2994 such pairs.

The author is apparently unaware of three recent papers by SUTTON,[1] CHERWELL[2] and SELMER & NESHEIM[3] [*MTAC*, v. 2, p. 210, 342]. This latter table agrees as far as it goes (200 000) with the one under review. The discrepancy noted between Sutton and Selmer & Nesheim in *MTAC*, v. 2, p. 342, is thus to be blamed on Sutton, who omits two prime pairs between 70 000 and 80 000, one between 90 000 and 100 000 and one between 120 000 and 130 000.

On p. 58 the author gives a rather incomplete list of errata in KRAITCHIK's list of primes to 300 000. Missing errata,[5] however, do not happen to influence the author's table.

D. H. L.

[1] C. S. SUTTON, "An investigation of the average distribution of twin prime numbers," *Jn. Math. Phys.*, v. 16, 1937, p. 1–42.

[2] LORD CHERWELL, "Note on the distribution of the intervals between prime numbers," *Quart. Jn. Math.*, Oxford s., v. 17, 1946, p. 46–62.

[3] E. S. SELMER & G. NESHEIM, "Tafel der Zwillingsprimzahlen bis 200 000," K. Norske Videnskabers Selskab, Trondhjem, *Forhandlinger*, v. 15, 1942, p. 95–98.

[4] M. KRAITCHIK, *Recherches sur la Théorie des Nombres*, v. 1, Paris, 1924.

[5] On page 156 of the reviewer's *Guide to Tables in the Theory of Numbers*, Washington, 1941, is given what was hoped to be a complete list of errata in Kraitchik's list of primes. Tietze, however, notes two additional errata, namely:

for	$p = 252141$	read	$p = 252143$	
for	$p = 297671$	read	$p = 297971$	[*MTAC*, v. 2, p. 313]

734[F].—J. TODD, "A problem on arctangent relations," *Amer. Math. Monthly*, v. 56, 1949, p. 517–528.

There are two tables giving the representation of arctangents of integers and rationals as linear combinations with integral coefficients of arctangents of integers. The numbers

$$\text{(S)} \qquad 3, 7, 8, 13, 17, 18, 21, 30, \cdots$$

have the property that, if m be one of them, arctan m can be expressed in terms of arctangents of integers $<m$ and not in (S). Thus

$$\arctan 18 = 3 \arctan 1 - 2 \arctan 2 + \arctan 5.$$

The utility of such representations has been pointed out by J. C. P. MILLER [*MTAC*, v. 2, p. 62–63, 147–148] in connection with the preparation of a table of the Gamma function of a complex variable. The study of such relations goes back to GAUSS. The numbers of the set (S) are called reducible.

Table I (p. 525) lists all reducible integers not exceeding 342 and gives for each of these 100 numbers n the expression of arctan n in terms of the arctangents of irreducible integers. There is also given an auxiliary integer c_n permitting an immediate passage from an arctangent to an arccotangent relation, where c_n is the new coefficient of arccot 1. Thus for $n = 18$, $c_n = 1$ and

$$\operatorname{arccot} 18 = \operatorname{arccot} 1 - 2 \operatorname{arccot} 2 + \operatorname{arccot} 5.$$

Two errata have been supplied by the author

$n = 183$	*for*	$c_n = 1$	*read*	$c_n = 0$
	for	$4(1) + (3)$	*read*	$7(1) - (2)$
$n = 307$	*for*	$c_n = 2$	*read*	$c_n = -2$.

Table II gives for each prime $p \leqslant 409$ of the form $a^2 + b^2$ (i.e., for 2 and for all primes of the form $4k + 1$) expansions of arccot (a/b), $(a > b)$ as a linear combination of arctangents of irreducible integers. In addition there are given the numbers c for converting to arccotangents. With each p is given also the least positive integer n_p for which $1 + n^2_p = mp$, the quotient m is given in terms of its prime factors.

D. H. L.

735[G].—ALBERT SADE. *Sur les Chevauchements des Permutations*. Published by the author, Marseille, 1949. 8 p.

The tables (for $n \leqslant 12$) are of various classifications of the permutations of n distinct elements associated with the foldings of a linear strip of postage stamps which are characterized as being without "chevauchements." A chevauchement is an interlacing of number pairs a, $a + 1$, and b, $b + 1$ for a, b both odd or both even. Also, for $n \leqslant 8$, the permutations of n distinct elements with one element fixed are tabled according to the number of such interlacings.

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736[I].—R. E. Greenwood. "Numerical integration for linear sums of exponential functions," *Annals Math. Stat.*, v. 20, 1949, p. 608–611.

The author considers numerical integration over a finite range, using $n + 1$ evenly spaced abscissae. He introduces coefficients which would make the integration exact if the integrand were a linear combination of functions of the form exp (jx), where j runs from 0 to n (positive case), or from $-m$ to m, $2m = n$ (symmetric case). For both cases, for $n = 1(1)6$, the coefficients are given to from 5D to 9D; about 45 values in all. He compares the results with the usual Newton-Cotes method, for $n = 4$, and finds that his coefficients "compare favorably" with Newton-Cotes for the functions $1/(x + 3)$, $\exp(-x^2)$, $x \exp(x)$, x^6, and $\exp(2.2x)$.

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737[I, L].—H. E. Salzer & Ruth Zucker, "Table of the zeros and weight factors of the first fifteen Laguerre polynomials," Am. Math. Soc., *Bull.* v. 55, 1949, p. 1004–12.

For the first fifteen Laguerre polynomials, the zeros $x_i^{(n)}$ are given to 12 decimal places, the weight factors

$$\alpha_i^{(n)} = [n!/L_n'(x_i^{(n)})]^2/x_i^{(n)}$$

and values of $\alpha_i^{(n)} \exp\{x_i^{(n)}\}$ to 12 significant places. The chief use of the table is for numerical integration in the semi-infinite range by means of the mechanical quadrature formula

$$\int_0^\infty f(x)dx = \sum_{i=1}^{n} \alpha_i^{(n)} \exp\{x_i^{(n)}\} f(x_i^{(n)})$$

which is exact when $e^x f(x)$ is a polynomial of degree not exceeding $2n - 1$.

A. E.

738[K].—H. J. Godwin, "On the estimation of dispersion by linear systematic statistics," *Biometrika*, v. 36, 1949, p. 92–100.

Let $x_1 \leqslant x_2 \leqslant \cdots \leqslant x_n$ be the order statistics of a random sample of n from a normal population with cumulative distribution function

$$F(x) = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{x} \exp(-\tfrac{1}{2}t^2)dt,$$

and let $y_i = x_{i+1} - x_i$ $(i = 1, 2, \cdots, n - 1)$. Table 2 gives $E(y_i)$ and $E(y_i y_j)$ to 5 or more decimals for $n = 2(1)10$. The table of $E(y_i y_j)$ supersedes an equivalent table to 2 decimals by Hastings, Mosteller, Tukey, & Winsor [RMT 740]. To calculate Table 2, $\psi(i)$ and $\psi(i, j)$ defined by

$$\psi(i) = \int_{-\infty}^{\infty} F^i(x)[1 - F(x)]^i dx$$

$$\psi(i, j) = \int_{-\infty}^{\infty} F^i(x) \int_{x}^{\infty} [1 - F(y)]^j dy dx$$

were used; these are tabled to 10 decimals in Table 1, for $i = 1(1)5$ and $j = i(1)10 - i$. Suppose next that the differences y_i are defined as above but with $F(x)$ replaced by $F_\sigma(x)$, where $F_\sigma(\xi) = F(\xi/\sigma)$.

Table 3 gives to 5 or more decimals the coefficients α_i which minimize the variance of $d = \sum_{i=1}^{n-1} \alpha_i y_i$ subject to the condition that $E(d) = \sigma$, for $n = 2(1)10$. Table 4 lists the efficiencies (calculated as inverse ratios of variances) of the unbiased estimates of σ formed from the following statistics: the "best" linear estimate d, the sample mean deviation from the mean, the sample mean deviation from the median, and $x_{n-k+1} - x_k$ for $k = 1(1)[\frac{1}{2}n]$, $n = 2(1)10$; the efficiencies are given to tenths or hundredths of a per cent.

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739[K].—E. J. GUMBEL, "Probability tables for the range," *Biometrika*, v. 36, 1949, p. 142–148.

The author is concerned with the asymptotic distribution of the range in a sample of n independent equally distributed random variables (that is, the difference between the largest and the smallest values in the sample). The desired distribution density is given by $\psi(R) = 2e^{-R}K_0(2e^{-1/2R})$, where $K_0(x)$ is the Bessel function. On p. 145 we find the values of $\psi(R)$ and

$$\Psi(R) = \int_{-\infty}^{R} \psi(x)dx.$$

The range is $-3.2(.1)10.6$. The number of decimals drops from 8 at the tail ends ($R < -2.9$ and $R > 9$) to 5 in the central part (about $-1.1 < R \leqslant 3$). [See also *MTAC*, v. 4, p. 21.]

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740[K].—CECIL HASTINGS, JR., FREDERICK MOSTELLER, J. W. TUKEY & C. P. WINSOR, "Low moments for small samples: a comparative study of order statistics," *Annals Math. Stat.*, v. 18, 1947, p. 413–426.

This contains tables of the means, variances, and covariances of the order statistics in samples of size $\leqslant 10$ from a normal universe, a rectangular universe, and a special universe with very long tails. For the normal universe and the special universe values computed from asymptotic formulas are also given. Means are given to 5D in all cases and are believed to be accurate to one unit in the fifth decimal. For the normal universe standard deviations are given to 5D with a maximum error of 2 or 3 units in the fifth decimal, while variances and covariances are given to 2D, with a possible error of one unit in the second decimal (except in one or two cases in which the error may be two units). Variances and covariances for the other two cases are given to 5D and are thought to be accurate to the places given. In addition the correlation coefficients in all cases are given to 2D.

C. C. C.

741[K].—PALMER O. JOHNSON, *Statistical Methods in Research*, xvi + 377 p. New York, Prentice Hall, 1949, 15 × 22.7 cm. Price $5.00.

This book has five tables in an appendix. Tables I, II, III, and IV are standard tables of areas and percentiles for the normal, t, χ^2, and F distributions. These tables occur in most modern applied statistics books or tables.[1]

Table V contains first and fifth percentile values of the distribution of the NEYMAN-PEARSON L_1 statistic which is used to test the hypothesis that k normally distributed populations have equal variances. This test was designed for samples of sizes $n_1, n_2, \cdots, n_k$. These tables were constructed for the case $n_1 = n_2 = \cdots = n_k = n$, and have as arguments $k = 2(1)10(2)30$, $n = 2(1)10, 12, 15, 20, 30, 60, \infty$. They are reproduced from the *Statistical Research Memoirs* v. 1 edited by J. NEYMAN and E. S. PEARSON, and have not been widely reproduced elsewhere. These *Memoirs* are no longer easily available.

Examples of the use of several other tables are included in the text. References are made to those tables, but they have not been printed in the text. Some would not be easily available to most readers of this book. For example page 102, reference 22, is to a table in *Sankhya*, v. 4, 1938.

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[1] For example, see:
R. A. FISHER & F. YATES, *Statistical Tables for Biological Agricultural and Medical Research*, Edinburgh, 3rd ed., 1948 [*MTAC*, v. 3, p. 360–361] from which Tables II and III were reproduced.
G. W. SNEDECOR, *Statistical Methods Applied to Experiments in Agriculture and Biology*, Ames, Iowa, 1940, from which Table IV was reproduced. For errata see *MTAC*, v. 1, p. 85–86.

742[K].—P. B. PATNAIK, "The non-central χ^2- and F-distributions and their applications," *Biometrika*, v. 36, 1949, p. 202–232.

A noncentral chi-square variable, denoted by χ'^2, with n degrees of freedom (d.f.) and parameter χ may be defined as follows: Let x_i $(i = 1, \cdots, n)$ be n independent standard normal deviates, let a_i $(i = 1, \cdots, n)$ be n constants, and define $\chi'^2 = \sum_{i=1}^{n} (x_i + a_i)^2$. The distribution of χ'^2 is known to depend only on n and $\lambda = \sum_{i=1}^{n} a_i^2$. Noncentral F, denoted by F', with ν_1 and ν_2 d.f. and parameter λ may be defined as a random variable distributed like $(\nu_2\chi'^2)/(\nu_1\chi^2)$, where χ'^2 is a noncentral chi-square variable with ν_1 d.f. and parameter λ, χ^2 is a (central) chi-square variable with ν_2 d.f., and χ'^2 and χ^2 are statistically independent. Various approximations to the χ'^2 and F'-distributions are considered. There are seven small tables, six of which compare these approximations with exact values, one of which (Table 6) gives the power of a chi-square test at the 5% significance level, calculated from one of the approximations to the χ'^2-distribution. Tables similar to Table 6 but more extensive, for the 1% and 5% levels, and not based on approximations to the distributions, have been published recently by EVELYN FIX.[1] The

following approximations are considered for χ'^2 with n d.f. and parameter λ: (i) χ'^2 is approximated by χ^2/K, where χ^2 is a (central) chi-square variable with ν d.f. (in general, fractional), and the constants ν and K are determined by fitting the first two moments; (ii) normal approximation; (iii) and (iv) two different series for the cumulative distribution function of χ'^2 in each of which (i) contributes the leading term. F' with ν_1 and ν_2 d.f. and parameter λ is approximated as F/K, where F has the (central) F-distribution with ν and ν_2 d.f., K and ν being fitted by the first two moments. The result is the same as though the above approximation (i) were used for the χ'^2 in the numerator of F'.

HENRY SCHEFFÉ

[1] EVELYN FIX, "Tables of noncentral χ^2," University of California, *Publications in Statistics*, v. 1, 1949, no. 2, p. 15–19.

743[L].—W. R. ABBOTT, "Evaluation of an integral of a Bessel function," *Jn. Math. Physics*, v. 28, 1949, p. 192–194.

The integral $v(t, n) = 2n \int_0^t u^{-1} J_{2n}(\gamma u) du$ occurs in the theory of transmission lines.[1] The expression of v as a finite combination of Bessel functions is known, but the two expansions obtained in this paper,

$$v = 1 - \frac{2}{\gamma t} \sum_{k=1}^{n} (2k - 1) J_{2k-1}(\gamma t)$$

and

$$v = 1 - \sum_{p=0}^{n-1} \frac{(-)^p (2n - p - 1)!}{p!(n - p - 1)!(n - p)!} \Lambda_{n-p}(\gamma t)$$

are better suited for numerical computation. The coefficients of the latter expansion are given numerically for $n = 1(1)9$. For $n > 9$ the first expansion is more useful.

A. E.

[1] See for example, M. F. GARDNER & J. L. BARNES, *Transients in Linear Systems, Studied by the Laplace Transformation*, v. 1, New York, 1942, p. 310–317.

744[L].—MILTON ABRAMOWITZ, "Asymptotic expansions of Coulomb wave functions," *Quart. Appl. Math.*, v. 7, 1949, p. 75–84.

The author considers the differential equation

$$\frac{d^2y}{d\rho^2} + (1 - 2\eta/\rho)y = 0,$$

and gives series-expansions of its solutions for different relative magnitudes of ρ and η (both real), which may be used for computational purposes. Starting with WHITTAKER's integral representations of the confluent hypergeometric function, he defines two solutions y_1 and y_2 of the differential equation which are related to Whittaker's confluent hypergeometric function by the relations:

$$-2i\Gamma(1 + i\eta)y_1 = W_{i\eta,-\frac{1}{2}}(2i\rho)$$
$$2i\Gamma(1 - i\eta)y_2 = W_{-i\eta,-\frac{1}{2}}(-2i\rho).$$

He obtains asymptotic series for these solutions in the case that η is bounded and $\rho \to \infty$. These series are used to compute a table of the first three zeros of the function

$$\rho\Phi_0(\rho, \eta) = e^{\pi\eta}y_1 + y_2$$

for $\eta = 0(.5)3$. Three other expansions for $\rho\Phi_0(\rho, \eta)$ are obtained, one for the case $\rho < 2\eta$, one for $\rho > 2\eta$, and one for $2\eta < 1$ and ρ bounded. For bounded ρ and $\eta \to \infty$, an approximation in terms of Bessel functions is given.

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745[L].—C. G. DARWIN, "On Weber's function," *Quart. Jn. Mech. Appl. Math.*, v. 2, 1949, p. 311–320.

WEBER's equation is the differential equation

$$\frac{d^2u}{dz^2} + (n + \tfrac{1}{2} - \tfrac{1}{4}z^2)u = 0.$$

The case $z = x\sqrt{i}$, $n = -\tfrac{1}{2} + ia$ (x, a real) arises in various wave problems, and is studied here. Instead of utilizing the theory of confluent hypergeometric functions, of which WEBER's function is a particular instance, the author develops his theory *ab initio*. The even and odd solutions of the equation are unsuited because they are nearly proportional to each other for large x. The solutions selected for numerical tabulation now in progress at Scientific Computing Service, Ltd. (London) on behalf of the (British) National Physical Laboratory are

$$u_I = 2^{-\frac{1}{2}}\{(G_1/G_3)^{\frac{1}{2}}u_0 + (2G_3/G_1)^{\frac{1}{2}}u_1\}$$
$$u_{II} = 2^{-\frac{1}{2}}\{(G_1/G_3)^{\frac{1}{2}}u_0 - (2G_3/G_1)^{\frac{1}{2}}u_1\},$$

where

$$G_1 = |\Gamma(\tfrac{1}{4} + \tfrac{1}{2}ia)|, \qquad G_3 = |\Gamma(\tfrac{3}{4} + \tfrac{1}{2}ia)|,$$

and u_0 and u_1 are the even and odd solutions normalized to values 1 and x for small x. For positive a, u_I resembles an exponential function near the origin, and becomes oscillatory when $|x| > 2a^{\frac{1}{2}}$; for negative a, the function is oscillatory throughout. Convergent power series for small x, and asymptotic series for large x are given, both valid for fixed a. Other asymptotic expansions, valid for large a, are also developed.

Weber's equation, with real x and a, has also been investigated by MAGNUS,[1] and by CHERRY.[2]

A. E.

[1] W. MAGNUS, Deutsche Math.-Ver., *Jahrsb.*, v. 50, 1940, p. 140–161.
[2] T. M. CHERRY, Edinburgh Math. Soc., *Proc.*, s. 2, v. 8, 1948, p. 50–65.

746[L].—G. H. GODFREY, "Diffraction of light from sources of finite dimensions," *Australian Jn. Sci. Res.*, s. A, v. 1, 1948, p. 1–17.

The paper contains tables concerned with the diffraction of light at rectangular and circular apertures. In particular Table 1 (p. 7) is a table of

$$I(x) = [\mathrm{Si}(2x) - x^{-1}\sin^2 x]/\pi$$

for $x = [0(.1)15(.5)34.5; 5D]$.

There are also tables of the differences

$$I(x + 2.4\pi) - I(x) \quad \text{for} \quad x/\pi = -1.2(.1)1.3$$
$$I(x + 3.4\pi) - I(x) \quad \text{for} \quad x/\pi = -1.7(.1)0$$

to 5D. Table 4 (p. 13) gives 4D values of

$$I_1(x) = \int_0^x [H_1(2t)/(\pi t^2)]dt$$

for $x = 0(.1)15$, where

$$H_1(U) = \pi^{-1}\left\{2 + \int_0^\pi \sin(u \sin\theta - \theta)d\theta\right\}$$

is STRUVE's function of order unity.

Table 5 (p. 15) gives the difference

$$I_1(x + 8.4) - I(x) \quad \text{for} \quad x = [-4.2(.1) - 2.8; 4D].$$

D. H. L.

747[L].—E. T. GOODWIN & J. STATON, "Table of $J_0(j_{0n}r)$," *Quart. Jn. Mech. Appl. Math.*, v. 1, 1948, p. 220–224.

Five decimal values of $J_0(j_{0n}r)$ are tabulated for $n = 1(1)10$ and $r = 0(.01)1$; j_{0n} being the nth positive zero of the BESSEL function of the first kind J_0. These values were calculated to seven decimal places by interpolation from the B. A. *Math. Tables*, v. vi, of Bessel functions; they were then differenced in the r direction on the National Accounting Machine as a check, and all doubtful roundings off were examined. The tables should be useful in the numerical discussion of potential problems with axial symmetry.

A. E.

748[L].—D. R. HARTREE, "The tabulation of Bessel functions for large argument," Cambridge Phil. Soc., *Proc.*, v. 45, 1949, p. 554–557.

The use of auxiliary functions in tabulating, to simplify interpolation, is well known. The author pleads for the use of auxiliary independent variables for the same purpose. As an illustration, he shows that linear interpolation in a suitably constructed table of only 41 entries should be sufficient to give $x^{\frac{1}{2}}J_0(x)$ and $x^{\frac{1}{2}}Y_0(x)$ from $x = 5$ to ∞ with an uncertainty of one unit in the seventh decimal. The auxiliary variable is x^{-2}, and the auxiliary functions are either $P(x)$ and $Q(x)$ defined by

$$J_0 + iY_0 = (2/\pi x)^{\frac{1}{2}}(P + iQ)e^{i(x-\pi/4)},$$

or $R(x)$ and $x\psi(x)$ defined by

$$J_0 + iY_0 = (2/\pi x)^{\frac{1}{2}}Re^{i(x-\pi/4+\psi)}.$$

Tables are given for $(2/\pi)^{\frac{1}{2}}P$ and $(2/\pi)^{\frac{1}{2}}Q$, and also for $(2/\pi)^{\frac{1}{2}}R$ and $x\psi$ for $x^{-2} = 0(.01).05$.

A. E.

749[L].—Harvard University, Computation Laboratory, *Annals*, v. 13: *Tables of the Bessel Functions of the First Kind of Orders Sixty-four through Seventy-eight.* Cambridge, Mass., Harvard University Press, 1949, x, 566 p. 20 × 26.6 cm. Offset print. Price $8.00.

A summary of reviews in *MTAC* of earlier published volumes of the *Annals* is given in RMT 711. The volume under review is the eleventh, in the monumental Harvard series of tables of Bessel functions of the first kind, giving $J_n(x)$ for $n = 64(1)78$, $x = [0(.01)99.99; 10D]$, but prior to $x = 37.16$ all values of $J_n(x)$ to 10D are zero. $J_{78}(48.78)$ is the first significant value of this function.

All values in this table have not been previously published. Two more volumes in preparation will deal with the next 22 orders, and give also the values of $J_n(100)$ for $n = 0(1)100$.

R. C. A.

750[L].—Harvard University, Computation Laboratory, *Annals*, v. 21: *Tables of the Generalized Exponential-Integral Functions.* Cambridge, Mass., Harvard Univ. Press, 1949, xxv, 416 p., 20 × 26.6 cm., $8.00.

The functions tabulated in this useful volume are

$$E(a,x) = \int_0^x (1 - e^{-u})u^{-1}dt, \qquad Es(a,x) = \int_0^x e^{-u}u^{-1}\sin u\,dt,$$

and

$$Ec(a,x) = \int_0^x (1 - e^{-u}\cos u)u^{-1}dt,$$

where $u = (a^2 + t^2)^{\frac{1}{2}}$. The integrals

$$\bar{E}(a,x) = \int_x^\infty e^{-u}u^{-1}dt \qquad \text{and} \qquad \int^\infty u^{-1}\cos u\,dt$$

can be evaluated in terms of elementary functions and the functions tabulated in this volume. All these functions are related to the generalized sine- and cosine-integral functions tables of which appeared in two earlier volumes of the same series[1] [*MTAC*, v. 3, p. 479–482]; they are also related (in special cases) to the exponential integral function by means of the relation

$$E(0,x) + \mathrm{Ei}(-x) = \log x + \gamma,$$

where log denotes the natural logarithm and γ is Euler's constant. These functions may also be regarded as incomplete (modified) Bessel functions, for instance

$$\bar{E}(a, \infty) = K_0(a).$$

Asymptotic representations (for large x) can be obtained by the usual method of integration by parts.

Generalized exponential integral functions occur in the solution of the wave equation when line sources are immersed in a dissipative medium, and were encountered in particular in antenna theory. Their tabulation was undertaken, on the Automatic Sequence Controlled Calculator, at the request of Ronold W. P. King. First the integrands were tabulated, and then the integrals computed by numerical integration, partly by Weddle's rule, and partly by two rules based on fifth degree polynomials. An estimate

of the error in the integrands and of that due to quadrature, together with a study of the difference sheets, led to an estimate of 3×10^{-8} for the maximum error in the integrals. The values, including differences, were rounded off to 6 decimals; an accuracy of 5.1 in the seventh decimal place is claimed for all numbers printed.

The volume contains a Preface by HOWARD AIKEN, an Introduction of four sections of which I "The generalized exponential-integral functions," II "Computation of the tables," III "Interpolation" were written by J. ORTEN GADD and THEODORE SINGER who also collaborated in preparing the coding and control tapes and supervising the calculation, and IV "Applications" by R. W. P. King and C. T. TAI. There are 8 tables each for a different value of the interval h. In each table both a and x run from 0 to $49h$ at steps of h. Each table consists of 50 pages of fifty lines each; each page contains, for a fixed value of a and for x running from 0 to $49h$, 6 decimal values of $E(a, x)$, $Es(a, x)$, and $Ec(a, x)$ together with first forward differences in both the a- and x-direction, except that differences are omitted when they can be found in the earlier parts of the volume. In the eight tables, h is respectively 0.001, 0.002, 0.005, 0.01, 0.02, 0.05, 0.1, 0.2.

All in all, this is a very useful and excellently arranged volume.

A. E.

[1] HARVARD UNIVERSITY COMPUTATION LABORATORY, *Annals*, v. 18, 19: *Tables of the Generalized Sine- and Cosine- Integral Functions*, Parts I and II, Cambridge, Mass., 1949.

751[L].—C. W. JONES, "On a solution of the laminar boundary-layer equation near a position of separation," *Quart. Jn. Mech. Appl. Math.*, v. 1, 1948, p. 385–407.

This paper is mainly concerned with coordinating the investigations of the problem mentioned in the title by S. GOLDSTEIN on the one hand, and by D. R. HARTREE on the other hand. Since rigorous proofs seemed too difficult, the discussion is largely computational.

Goldstein[1] expands the potential of the flow downstream in the form

$$\psi = \xi^2[f_0(\eta) + \xi f_1(\eta) + \xi^2 f_2(\eta) + \cdots].$$

The f_r satisfy differential equations of the form

$$f_r''' - \tfrac{1}{2}\eta^2 f_r'' + (\tfrac{1}{2}r + 2)\eta^2 f_r' - (r + 3)\eta f_r = G_r,$$

where G_r depends on $f_0, \cdots, f_{r-1}$, and suitable boundary conditions.

$$f_0 = \eta^3/6, \qquad f_1 = \alpha_1\eta^2, \qquad f_2 = \alpha_2\eta^2 - \alpha_1^2\eta^5/15.$$

The differential equation for f_r, for $r \geqslant 3$, cannot be integrated explicitly, and so Jones uses for f_3 a step-by-step numerical integration, starting with the asymptotic solution for large η. His Table 1 gives f_r/α_1^r, and the derivative of this function, for $r = 0, 1, 2, 3$ and $\eta = 0(.1)4$, to a varying number of decimal places. f_4 and f_6 contain an unknown constant, and in these cases only those parts of the functions, and of their derivatives, which are independent of that constant are tabulated, for the same η, in Table 2. There is also a Table 3 giving $\sum \xi^r f_r'$ as function of η for several values of $\alpha_1\xi$, but this table, as the author points out, is of less general application.

A. E.

[1] S. GOLDSTEIN, "On laminar boundary-value flow near a position of separation," *Quart. Jn. Mech. Appl. Math.*, v. 1, 1948, p. 43–69.

752[L].—GEORGETTE DE NOCKERE, *Tables Numériques des Polynômes de Legendre* $P_{n,0}(\cos\theta)$ *et des Fonctions Associées* $P_{n,j}(\cos\theta)$ *ainsi que de leurs Intégrales* P^* *jusqu'a* $n = 15$ *et* $j = 4$, *pour l'argument* θ *(colatitude) variant de degré en degré. Tableaux des latitudes et longitudes divisionnaires et valeurs des multiplicateurs pour le calcul des coefficients du développement en série de polynômes de Laplace par la méthode des compartiments équivalents, d'une fonction de deux variables indépendantes.* Acad. r. de Belgique, *Cl. d. Sciences, Mémoires*, v. 24, fasc. 4, publ. no. 1592, 1949, 166 p. 16.1 × 25 cm. Price 150 Belgian francs.

The first table, 5D, Δ (p. 12–43), is of $P_{n,0}$ and of $P^*_{n,0} = \int P_{n,0}(\cos\theta)d\cos\theta$, for $n = 0(1)15$, $\theta^\circ = 0(1)90$. Here $P_{0,0} = 1$; $P_{1,0} = \cos\theta$, $P^*_{1,0} = \frac{1}{2}\cos^2\theta$; $P_{2,0} = \frac{1}{4}(3\cos 2\theta + 1)$, etc.

The second main table, 5D, Δ, (p. 44–151), is of $P_{n,j}$ and of

$$P^*_{n,j} = \int P_{n,j}(\cos\theta)d\cos\theta$$

for $n = 1(1)15$, $j = 1(1)4$, $\theta^\circ = 0(1)90$. Here

$$P_{1,1} = \sin\theta,\ P^*_{1,1} = \tfrac{1}{4}(\sin 2\theta - 2\theta) + \tfrac{1}{4}/\pi;\ \text{etc.}$$

The constants of integration are always chosen such that for $\theta = 90^\circ$, $P^*_{n,j} = 0$. Because of this condition the values for $\int P_{n,j}(\cos\theta)d\cos\theta$ given by G. PRÉVOST, *Tables de Fonctions Sphériques et de leurs Intégrales*. . . . Bordeaux and Paris, 1933, p. 153*–155*, have to be checked for appropriate constants of integration. In calculating the integrals, computations were made either directly, or with the aid of the following recurrence relations of LIÉNARD:[1]

$$\begin{aligned}(n+2)(n+1-j)\int &P_{n+1,j}(\cos\theta)d\cos\theta \\ &= (n-1)(n+j)\int P_{n-1,j}(\cos\theta)d\cos\theta \\ &\qquad - (2n+1)\sin^2\theta P_{n,j}(\cos\theta).\end{aligned}$$

$$\begin{aligned}(j-1)\int &P_{n,j+1}(\cos\theta)d\cos\theta \\ &= (j+1)(n+j)(n+1-j)\int P_{n,j-1}(\cos\theta)d\cos\theta \\ &\qquad + 2j\sin\theta P_{n,j}(\cos\theta).\end{aligned}$$

The original contributions of Miss Nockere consist in computing:

(i) the values of $P_{n,j}(\cos\theta)$ for $n = 9(1)15$, $j = 1(1)4$, by the formula,

$$P_{n+1,j}(\cos\theta) = (2n+1)\sin\theta\, P_{n,j-1}(\cos\theta) + P_{n-1,j}(\cos\theta)$$

(ii) the values of $P^*_{n,0}$ for $n = 0(1)15$, by means of the classical formula,

$$(2n+1)\int P(\cos\theta)d\cos\theta = P_{n+1}(\cos\theta) - P_{n-1}(\cos\theta) + \text{const.}$$

(iii) the values of $P_{n,j}$, $n = 1(1)15$, $j = 1(1)4$.

The values of $P_{n,j}(\cos\theta)$ for $n = 0(1)8$, $j = 0(1)4$, were adapted from the 7–10D tables of TALLQVIST, (a) "Tafeln der Kugelfunctionen $P_n(\cos\theta)$,"

1905; (b) "Tafeln der abgeleiteten und zugeordneten Kugelfunctionen erster Art," 1906, Finska Vetenskaps-Societeten, *Acta*, v. 33, nos. 4; 9.

Since the Preface to the volume was written by Prévost, and dated 1938, it seems probable that these tables were computed some years ago. On the title page is the statement: "Impression décidée 4 mai 1948."

The final section (p. 153–166) is devoted to the subject matter referred to in the latter part of the title and is an elaboration of ideas developed in Prévost's work mentioned above.

R. C. A.

[1] A. Liénard, "Formules de récurrence pour les intégrales des fonctions adjointes des polynômes de Legendre," Acad. d. Sciences, Paris, *Comptes Rendus*, v. 196, 1933, p. 1773–1778 (there's a slip in the page reference here).

753[L].—J. Pachner, "Pressure distribution in the acoustical field excited by a vibrating plate." Acoust. Soc. Amer., *Jn.*, v. 21, 1949, p. 617–625.

Notations: z_{mn} is the n-th root of the equation

$$J_m(iz)J_m'(z) - J_m(z)J_m'(iz) = 0,$$

$A_{mn} = e_m J_m^2(z_{mn}) J_m^2(z_{mn}i) i^{2m}$, where $e_0 = 2$ and $e_m = 1$ for $m = 1, 2, \cdots,$
$B_{mn} = 2i^m J_m(iz_{mn}) J_{m-1}(z_{mn})/z_{mn}$
$C_{mn} = J_m(z_{mn})/(z_{mn} J_{m-1}(z_{mn}))$.

Table I: z_{mn} to 1 to 3 decimal places for $m = 0$ and $n = 0(1)4$, $m = 1$ and $n = 0(1)3$, $m = 2$ and $n = 0(1)3$, $m = 3$ and $n = 0(1)2$, $m = 4$ and $n = 0, 1$, $m = 5(1)7$ and $n = 0$.

Table II: A_{0n} to 4 significant figures for $n = 0(1)3$.

Table III: A_{mn} to 4 significant figures for $m = 1$ and $n = 0(1)3$, $m = 2$ and $n = 0, 1$, $m = 3(1)5$ and $n = 0$.

Table IV: B_{mn} to 4 significant figures for $m = 0, 1$ and $n = 0(1)3, m = 2$ and $n = 0, 1$, $m = 3(1)5$ and $n = 0$.

Table V: C_{mn} to 4 decimal places for $m = 0, 1$ and $n = 0(1)3$.

Table VI: $2mC_{mn}$ to 4 decimal places for $m = 0$ and $n = 0(1)3$, $m = 1$ and $n = 0, 1$, $m = 3(1)5$ and $n = 0$.

Table VII: Table of values where the partial acoustic pressure falls to zero.

There are also various graphs representing numerical values.

E. A.

754[L].—Charles H. Papas & Ronold King, "Input impedance of wide-angle conical antennas fed by a coaxial line," IRE, *Proc.*, v. 37, no. 11, 1949, p. 1269–1271.

This paper introduces the auxiliary functions $\zeta_n(x) = g_n(x) + ib_n(x)$, which are related to the spherical Hankel function of the second kind $h_n^{(2)}$ by the equation:

$$\zeta_n(x) = \frac{h_n^{(2)}(x)}{h_{n-1}^{(2)}(x) - nx^{-1}h_n^{(2)}(x)}$$

and gives values for the real and imaginary parts of $\zeta_n(x)$ in the form of curves covering the range $1 < x < 15$ and for g_1 to g_{20} and b_1 to b_{17}.

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755[L].—M. Rothman, "Table of $\int_0^x I_0(x)dx$ for 0(0.1)20(1)25," *Quart. Jn. Mech. Applied Math.*, v. 2, 1949, p. 212–217.

The function $f(x) = \int_0^x I_0(t)dt$, where $I_0(t)$ is the Bessel function of imaginary argument, is tabulated for x = [0(.1)20; 8S], and the function $e^{-x}f(x)$ is given to 9S for x = 15(1)25. In both tables modified second and fourth differences are given.

These tables extend considerably those previously published [*MTAC*, v. 1, p. 250, v. 3, p. 308].

D. H. L.

756[L].—V. V. Solodovnikov, "O primenenii trapetsoidalnykh chastotnych kharakteristik k analizu kachestva sistem avtomaticheskogo regulirovaniîa" [On the application of trapezoidal frequency characteristics to the analysis of the behavior of systems of automatic regulation], *Avtomatika i Telemekhanika*, v. 10, 1949, p. 362–376.

To aid in the treatment of the transform

$$g(t) = (2/\pi)\int_0^\infty x^{-1}f(x)\sin tx\,dx$$

the author tabulates (p. 370–371) the function

$$(2/\pi)\{\mathrm{Si}(kt) - (1-k)^{-1}[\mathrm{Si}(t) - \mathrm{Si}(kt) + t^{-1}(\cos t - \cos kt)]\}$$

for [k = 0(.05)1, t = 0(.5)26; 3D]. This function, which is the transform of a standard "trapezoidal function," is also graphed for k = .8, .9, 1. [cf. RMT **726**].

D. H. L.

757[L].—A. Ulrich, "Die ebene laminare Reibungschicht an einem Zylinder." *Arch. d. Math.*, v. 2, 1949, p. 37–41.

The functions appearing in the integration of the boundary layer equations according to the method of Blasius[1] and Howarth[3] satisfy certain nonlinear differential equations, of which the author gives explicitly those for the functions tabulated in his paper.

Table 1. Tables of f_1 and f_3 with first and second derivatives to 4 decimal places for η = 0(.1)4.

Table 2. Tables of g_5 and h_5 with first and second derivatives to 4 decimal places for η = 0(.2)4.

Table 3. Tables of g_7, h_7, k_7, g_9, h_9, k_9, j_9, q_9 with first and second derivatives to 3 decimal places for η = 0(.1)4.

Tables of f_1 and f_3 have been first given by Hiemenz.[4] Howarth[3] improved the values of f_3 and gave first approximations for g_5 and h_5, which were later improved by Frössling.[2] The functions with subscripts 7 and 9 are tabulated here for the first time. They were obtained by numerical integration, by Adams' method, of the nonlinear ordinary differential equations which they satisfy.

A. E.

[1] H. Blasius, "Grenzschichten in Flüssigkeiten mit kleiner Reibung," *Zschr. f. Math u. Phys.*, 56, 1908, p. 1.

[2] N. Frössling, "Verdunstung, Wärmübertragung und Geschwindigkeitsverteilung bei zweidimensionaler und rotationssymmetrischer laminaren Grenzströmung," Lund, Sweden, Univ., *Acta*, Afd. 2, *Årsskrift*, v. 36, 1940, p. 1–32.

[3] L. HOWARTH, "On the calculation of steady flow in the boundary layer near the surface of a cylinder in a stream," ARC *Report* no. 1632, 1934.

[4] K. HIEMENZ, "Grenzschichten an einem in einen gleichförmigen Flüssigkeitsstrom eingetauchten geraden Kreiszylinder," *Dinglers Polytechn. Jn.*, v. 326, 1911, p. 321.

758[L, V].—G. N. WARD, "The approximate external and internal flow past a quasi-cylindrical tube moving at supersonic speeds," *Quart. Jn. Mech. Appl. Math.*, v. 1, 1948, p. 225–245.

In the course of the work indicated in the title, the following two functions are used:—$W(z)$ which is the inverse LAPLACE transform of $1 - K_0(p)/K_1(p)$, and $V(z)$ which is the inverse Laplace transform of $-p^{-1}K_1(p)/K_1'(p)$, where K_n is the modified BESSEL function of the third kind. Five decimal values of both functions, for $z = 0(.2)10$ have been computed by the Admiralty Computing Service of Great Britain and are reproduced here together with values of $W(z)$ for $z = -1.8(.2)-.2$ computed by the author and Miss ROUSSAK at Manchester, England.

Cf. also British Admiralty Report SRE/ACS 89, 1945 [*MTAC*, v. 2, p. 294–295].

A. E.

759[V].—ZDENĚK KOPAL, *Tables of Supersonic Flow Around Cones of Large Yaw.* Technical Report no. 5, Massachusetts Institute of Technology, 1949. xviii + 125 p., 19.5 × 26.8 cm.

The present volume is the third in the well-known series of computations of supersonic flow past cones.[1] The title of these tables is somewhat misleading. The title "Tables of Supersonic Flow Around Cones of Large Yaw" compared with the previous volume "On Supersonic Flow Past Slightly Yawing Cones" suggests two limiting cases, one for large and one for small angles of yaw. This, however, is not the case and the present tables deal with the second order approximation, the previous one with the first order approximation in a development in powers of the yaw angle ϵ. Both tables are based on partly unpublished results of STONE.

The radial and normal velocity components u, v and the pressure and density p and ρ are written in the general form

$$f = f_0 + \epsilon \sum_{n=0}^{\infty} f_{1,n} \cos n\phi + \epsilon^2 \sum_{n=0}^{\infty} f_{2,n} \cos n\phi + \cdots,$$

where ϵ denotes the angle of yaw, ϕ the circumferential angle. The axes are fixed with respect to the flow; the origin is at the apex of the cone. The circumferential velocity component w, i.e., the component in the direction of increasing ϕ is given in the form

$$g = \epsilon \sum_{n=0}^{\infty} g_{1,n} \sin n\phi + \epsilon^2 \sum_{n=0}^{\infty} g_{2,n} \sin n\phi + \cdots.$$

In these series f_0; $f_{1,n}$ and $g_{1,n}$; $f_{2,n}$ and $g_{2,n}$ stand for the zero, first order and second order terms respectively. The cos and sin series are due respectively to the boundary conditions which also lead to the result that only $f_{1,1}$ and $g_{1,1}$ differ from zero in the first order and $f_{2,0}$, $f_{2,2}$ and $g_{2,0}$, $g_{2,2}$ in the second order terms.

Hence, the tables consist essentially of tabulated values for the $f_{2,0}$, $f_{2,2}$; $g_{2,0}$ and $g_{2,2}$ corresponding to the five variables u, v, w, p, ρ. The velocities are referred to C, the maximum velocity obtainable by isentropic expansion.

Values are given for 7 cone opening angles in the range from 10° to 50° total included apex angle. An additional short survey of the main results is included. The representation, print, arrangement, etc., is similar to the previous volumes and of very high quality.

In general four significant figures are given in the numerical results. It is stated that this required an accuracy of six digits in the zero approximation (TAYLOR-MACCOLL solution) and five digits in the first order (STONE) approximation. It is pointed out that this accuracy is greater than present-day experimental methods warrant but kept for future improvements in experimental technique. On this point the reviewer does not agree: The necessary accuracy of computations of perfect fluid solutions has a natural limit due to viscous effects. Viscosity is bound to have an effect for flow past yawing cones especially at larger angles of yaw. Hence, this point together with limitations due to possible convergence difficulties in the series in ϵ seem to govern the desired accuracy rather than experimental errors. The reviewer feels that the accuracy in these tables is very probably higher than necessary. Still it may be argued that a standard solution like the cone case should be carried out with too great an accuracy just in order to enable an observer to evaluate viscous effects. The main justification of computing the cone flow in such great detail is, of course, the fact that here one standard solution is provided for comparison with more approximate theoretical methods on one hand and for experimental exploration of viscosity effects on the other.

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[1] See *MTAC*, v. 3, p. 37–40, 197–198.

760[V].—T. Y. THOMAS, "Calculation of the curvature of attached shock waves," *Jn. Math. Phys.*, v. 27, 1949, p. 279–297.

In a previous paper[1] the author considered the well-known problem of supersonic flow past a pointed, curved, two-dimensional object to obtain an expression for the curvature of the attached shock-wave in terms of the curvature of the stream-lines immediately behind the shock. The present paper contains a numerical application of the previous result to obtain the ratio of curvature of shock-wave to curvature of stream-line at the vertex of the body. The expression for the curvature then involves only the standard ratios characterizing an oblique shock—density on the two sides of the shock, etc. The set of tables and graphs expressing the results of the calculation, therefore, are simply those for the much-discussed flow against a wedge with the one additional quantity, the curvature ratio, included. Consequently, it is surprising to be told in §3 of the paper that to the semi-vertex angle zero we must associate the shock inclined at 90° rather than at the well-known Mach angle arccsc M. One is presumably to infer therefore that the author considers either the larger shock-inclination branch, of the well-known double solution to the problem, to apply to physical reality—or that one has a discontinuous jump from one branch to the other in the neighborhood of vertex angle zero. (So far as the reviewer is aware, neither of these phenomena has been observed although attempts have been made at devising experimental conditions under which the former might occur.) The mathematical error in the paper arises from the author's statement of

the compression condition as:

(density behind shock)/(density before shock) > 1 rather than $\geqslant 1$.

The worst physical error would seem to be that of publishing the discussion accompanying the tables without first looking at some of the experiments to which presumably they are meant to be applied.

If one ignores the discussion of §3 of the paper, and substitutes for it the caution that for a given semivertex-angle one should choose the smaller of the two shock-angles, the tables and graphs of the paper should be useful for obtaining the initial curvature of the shock in the case of a curved, pointed, two-dimensional object such as an air-foil. The tabulation is in the form of 13 tables, each for a single Mach number ($M = 1.05, 1.08, 1.12, 1.18, 1.25, 1.35, 1.47, 1.63, 1.83, 2.12, 2.56, 3.24, 4.45$) with argument within the table being the shock-angle. There is included the additional useful table giving the relation between vertex-angle, shock-angle, and Mach number for the greatest vertex-angle with attached shock at given Mach number. The accompanying graphs exhibit curvature ratio and vertex-angle as functions of shock-angle.

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[1] T. Y. Thomas, "On curved shock waves," *Jn. Math. Phys.*, v. 26, 1947, p. 62–68.

MATHEMATICAL TABLES—ERRATA

In this issue references have been made to Errata in RMT **729** (Chowla & Todd), **733** (Tietze), **734** (Todd), **741** (Johnson); UMT **93** (Fukamiya).

168.—F. S. Carey, "Notes on the division of the circle," *Quart. Jn. Math.*, v. 26, 1893, p. 332–371.

Table IV, giving the coefficient of the 6-nomial sextic, has the following errata. This list is the result of a recalculation of the table.

p	*Coefficient of*	*For*	*Read*
61	x^0	−27	27
109	x^2	39	135
181	x^0	13565	1685
193	x^0	−5182	−5184
	x	1936	1744
229	x^0	−2103	187
241	x	594	580
373	x^3	381	380
397	x	4960	−5040
433	x^0	−130032	−1728
457	x	3561	3461
103	x^0	1773	1373
127	x	−977	−972
151	x^0	6547	6543
163	x^0	21323	5023
223	x	−3276	5644
	x^0	−71228	4592
331	x^0	84429	84427

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169.—NBSMTP, *Tables of Spherical Bessel Functions*, v. 1, New York, Columbia University Press, 1947 [*MTAC*, v. 2, p. 308–309].

In the table of $(\pi/2x)^{\frac{1}{2}}J_{\frac{1}{2}}(x)$ at $x = 7.45$

for .12340 32451 *read* .12342 32451.

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170.—NBSMTP, *Tables of the Exponential Function* e^x, New York, 1st ed. 1939, 2nd ed. 1947 [*MTAC*, v. 1, p. 438, v. 2, p. 314, v. 3, p. 173].

P. 188, at $x = 1.8784$, *for* $x = 1.9884$ *read* $x = 1.8784$; and *for* $e^x = 6.54302\ 76384\ 25706$ *read* 6.54302 76384 25796; *for* $x = 1.9885, 1.9886$ *read* $x = 1.8785, 1.8786$.

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171.—L. Schwarz, "Untersuchung einiger mit den Zylinderfunktionen nullter Ordnung verwandter Functionen", *Luftfahrtforschung*, v. 20, 1943, p. 341–372. [Translated by J. Lotsof, Cornell Aeronautical Laboratory, 1946.]

The following errors were found by differencing the tables and consequently the corrected values can be in error by at most two units in the last decimal place.

Function	λ	x	*for*	*read*
$J_c(\lambda,x)$	0.2	1.58	0.9_5297	0.995297
$J_c(\lambda,x)$	1.0	1.74	0.884505	0.884547
$N_c(\lambda,x)$	0.1	0.88	1.784990	1.784986
$N_c(\lambda,x)$	0.1	0.96	1.861997	1.861965
$N_s(\lambda,x)$	0.1	1.38	1.357677	1.357670
$N_c(\lambda,x)$	0.2	0.44	0.967949	0.966929
$N_c(\lambda,x)$	0.2	0.46	0.995782	0.995784
$N_c(\lambda,x)$	0.2	0.58	1.155928	1.153056
$N_s(\lambda,x)$	0.2	0.30	0.097532	0.097509
$N_s(\lambda,x)$	0.2	0.44	0.184975	0.184746
$N_s(\lambda,x)$	0.2	0.58	0.289383	0.288526
$N_s(\lambda,x)$	0.2	0.60	0.304306	0.304406
$N_c(\lambda,x)$	0.7	1.48	0.831510	0.831310
$N_s(\lambda,x)$	0.7	0.06	0.003768	0.004336
$N_s(\lambda,x)$	1.0	1.54	0.005461	0.005036
$C_c(\lambda,x)$	0.0	0.10	0.069995	0.070995
$C_s(\lambda,x)$	0.0	0.58	0.064846	0.064865
$J_s(\lambda,x)$	0.2	4.1	1.27530258	1.57530258
$N_c(\lambda,x)$	0.9	4.9	1.388321	1.388821

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172.—G. N. WATSON, "A Table of Ramanujan's function $\tau(n)$," London Math. Soc., *Proc.*, s. 2, v. 51, 1949, p. 1–13 [*MTAC*, v. 3, p. 468].

P.12, $n = 847$ *for* 38152 *read* 58152.

This typographical error was noted when Watson's table was put on punch cards and submitted to a series of checks. These cards are available in the UMT FILE.

D. H. L.

UNPUBLISHED MATHEMATICAL TABLES

EDITORIAL NOTE: Beginning with this volume we are starting a collection of unpublished mathematical tables to be known as the UMT FILE. Authors of tables which have no immediate prospect of publication are invited to submit copies for deposit in UMT FILE. Description of such tables will appear in UMT and photostat or microfilm copies will be supplied at cost to any reader of *MTAC*. Address tables or correspondence to D. H. LEHMER, 942 HILLDALE AVE., BERKELEY 8, CALIFORNIA.

90[F].—R. A. LIENARD, *List of primes of the form* $k \cdot 10^6 + 1$ *and* $k \cdot 10^7 + 1$ *for* $k < 1000$. Manuscript in the possession of the author and deposited in UMT FILE.

There are listed 117 primes of the form $k \cdot 10^6 + 1$ and 109 primes of the form $k \cdot 10^7 + 1$.

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91[K].—BALLISTIC RESEARCH LABORATORIES. Aberdeen Proving Ground, Md., *Probability Integral of Extreme Deviation From Sample Mean.*

$$H_n(x) = (n/2\pi(n-1))^{\frac{1}{2}} \int_0^x \exp(-t^2/2n(n-1))H_{n-1}(t)dt, \quad (H_1(x) = 1.)$$

For normally distributed and ranked variables $u_1 \leqslant u_2 \leqslant \cdots \leqslant u_n$, $H_n(nk)$ gives the probability that the extreme deviation from the sample mean, i.e., $u_n - \bar{u}$ or $\bar{u} - u_1$ will not exceed k times the population standard deviation for random samples of size n, for $n = 1(1)25$. Each function is tabulated until it becomes sensibly unity. The interval of tabulation for x is as follows:

.1 for $n = 2(1)7$,	.4 for $n = 12, 13$,
.2 for $n = 8(1)11$,	.8 for $n = 14(1)25$.

The accuracy is 7D.

The work was done on the ENIAC under the direction of J. V. HOLBERTON.

92[K].—BALLISTIC RESEARCH LABORATORIES, Aberdeen Proving Ground, Md., *Binomial Probabilities.*

$$I_p(c, n-c+1) = \frac{1}{B(c, n-c+1)} \int_0^p x^{c-1}(1-x)^{n-c}dx.$$

The probability of c or more successes in n trials is given by $I_p(c, n - c + 1)$, where p is the chance of a success in one trial.

$$n = 1(1)200, \quad c = 0(1)n, \quad p = 0(.01)1;\ 7D$$

The work was done on the IBM Relay Calculator under the direction of M. LOTKIN.

93[L].—FASANORI FUKAMIYA, *Mathematical Studies of the High Frequency*, Photostat filed in the Japanese Section of the Board of Trade, Technical Information and Documents Unit. Reference BIOS/JAP/DOC/1550.

This paper contains a certain amount of descriptive text followed, it is stated, by 5 appendices.

Appendices I and II consist of tables and graphs of certain functions (which are not of any interest being purely "ad hoc") connected with radiation from a sectoral horn.

Appendix IV is stated to be a table of "zero points of the derivative of the Associated Legendre function." In actual fact Appendix IV consists of a series of graphs of some unknown function.

Appendix V is the most interesting part of the paper. This consists of tables of $J_n(z)$ as follows:

$$n = 1, \quad z = [0(.1)3;\ 7D], \quad n = 2(1)10, \quad z = [0(.1)10.7;\ 7D],$$
$$n = 11(1)20, \quad z = [0(.1)6;\ 7D], \quad n = 11(1)20, \quad z = [6.4(.1)10;\ 5D].$$

(no explanation is given for the absence of $J_n(z)$ for $n = 11(1)20$, $z = 6.1(.1)6.3$.)

The tables were compared as far as possible with WATSON'S Bessel Functions[1] and, apart from differences of 2 or less in the last place, the following discrepancies occur.

	Watson	Fukamiya
$J_2(2.3)$	.1799789	.1899789
$J_3(4.3)$	.4333147	.4333470
$J_5(1.0)$	.0002498	.0002475

It is further stated that tables of Fresnel Integrals for argument s 0–50 have been calculated, but were "destroyed in the event."

Unfortunately the reproduction of the copy seen was poor, only the three pages of text were typewritten, the tables being written. In addition appendix III was missing and no date was given for the work, nor was there any indication of where Fukamiya worked.

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A comparison with the forthcoming B. A., *Mathematical Tables*, v. X revealed many errors; these are often only a unit or two in the last figure, but many are gross errors. Some effect several values of n for the same x and are obviously computational. It is noteworthy that errors are fewer in the range covered by Watson's table[1] and are almost all in the end figure, yet

agreement with Watson is not complete, and a value known to be in error in Watson is correct in the table under review.

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[1] G. N. WATSON, *Treatise on the Theory of Bessel Functions*, Cambridge, 1922, second ed. 1944 [*MTAC*, v. 2, p. 49–51.]

94[V].—BALLISTIC RESEARCH LABORATORIES, Aberdeen Proving Ground, Md. *Supersonic Flow past Cone Cylinders.* [See Note **113.**]

AUTOMATIC COMPUTING MACHINERY

Edited by the Staff of the Machine Development Laboratory of the National Bureau of Standards. Correspondence regarding the Section should be directed to Dr. E. W. CANNON, 225 Far West Building, National Bureau of Standards, Washington 25, D. C.

TECHNICAL DEVELOPMENTS

Characteristics of the Institute for Numerical Analysis Computer

In January 1949, members of the staff of the Institute for Numerical Analysis[1] began the development and construction of a high-speed electronic digital computer. As of December 1, 1949, the central computer was approximately eighty per cent completed. The group responsible for the building of this machine is composed of, besides the author, three engineers, three junior engineers, and four technicians. In addition, one mathematician is assigned to the coding and programming of problems to be run on the machine.

Information is stored and processed in the computer in units called words, a word consisting of 41 binary digits. This word length is determined by the number of words which can be stored in the computer's high-speed memory.

Words in the machine sense may represent (1) numerical information, (2) instructions to the computer, and (3) alphabetic information.

In the case of numerical information, one binary digit of a word is used for the sign and 40 binary digits are available for numerical data. Numbers are stored in the memory as absolute value and sign. In the arithmetic unit, negative numbers may be converted to complementary[2] form to keep the operational algorithms relatively simple. Thus, negative numbers involved in addition, subtraction, and compare are complemented upon arrival in the arithmetic unit. In the multiplication, extract, input, and output commands, negative numbers are not complemented.

Numbers may be represented in many different ways. For example, a word may represent a signed-binary number lying somewhere between -2^{40} and $+2^{40}$, or the binary point may be ahead of the most significant digit in which case a word lies in the range -1 to $+1$; the built-in arithmetic operations handle numbers in either of these forms. The word may, on the other hand, represent a signed-ten-decimal-digit number where each decimal digit is represented as a four-digit-binary number. A floating representation may

be used where the first digit represents the sign, the next eight digits represent the exponent b, and the next 32 digits represent the significant digits of the number in binary form. A floating decimal representation may also be used giving numbers of eight significant digits ranging in absolute value from about 10^{-50} to 10^{+50}. More than one word can be used to represent a number to effect much greater precision or range of values.

Floating operations have been coded (as will be explained later in this paper) to provide for the addition, subtraction, multiplication, or division of two numbers of the form $a \cdot 2^b$ (with a and b stored in the same address). All four floating operations involve about 87 instructions. These floating operations are performed in about 3, 3, 6, and 14 milliseconds, respectively. Compare these times with 64, 64, and 384 *micro*-seconds, the times required for doing ordinary binary addition, subtraction, and multiplication.

Instructions are subclassified into three classes. These three classes are *command words*, *control words*, and *code words*.

The *command words*, commonly called commands, are explicitly "understood" and "obeyed" by the computer. A command causes the computer to perform a specific operation and gives the necessary information about where to get the needed data and what to do with the results. At present, the command system used by the Institute's computer consists of a set of thirteen commands.[3] Eight of the thirteen are what might be termed basic commands; the other five are variations of these eight. Such a command system is known as a four-five address code, with four addresses generally in the command word and the fifth address in a control counter. The function, or operation, of a particular command is denoted by F, the four addresses of the command by α, β, γ, and δ, and the fifth address by ϵ which normally determines the address of the next command to be obeyed.

The size of the memory determines the length of each address which, in the case of a 512 word memory, is nine binary digits. Thus, 36 binary digits are used to denote the four addresses, α, β, γ, and δ. Four digits are used to denote F; of these four, three are used to define the eight basic command words, and one is used to denote modifications of the eight. There is one spare digit in the command word.

The thirteen operations, or functions, of the Institute's computer, as well as the meanings of the addresses for the various command words, are given in Table 1.

In order to change automatically the course of operation in the calculator when certain bounds have been reached,[4] there are conditional and unconditional transfers of control commands. A conditional transfer is accomplished with compare commands. An unconditional transfer is accomplished with certain special commands (A_1, S_1, and M_1), wherein the fourth address, δ, determines the next command.

In the table, special compare might well be called absolute compare in that the absolute values of the numbers are compared. Since the result of the subtraction in the compare operation is put back into the memory, this command can be used to obtain the absolute value of a number in one operation by comparing the number with zero. The compare command can also be used for an operation called tally, as follows: Assume that it is desired to repeat a routine fifty times stored in memory locations 31 to 37, inclusive.

TABLE 1

Meanings of the Addresses for the Commands

Command	α	β	γ	δ	F
Add	Address of Augend	Address of Addend	Address of Sum	Address of Next Command if Overflow	A
Special Add	Address of Augend	Address of Addend	Address of Sum	Address of Next Command	A_1
Subtract	Address of Minuend	Address of Subtrahend	Address of Difference	Address of Next Command if Overflow	S
Special Subtract	Address of Minuend	Address of Subtrahend	Address of Difference	Address of Next Command	S_1
Multiply	Address of Multiplier	Address of Multiplicand	Address of Product Rounded Off		M
Special Multiply	Address of Multiplier	Address of Multiplicand	Address of Product Rounded Off	Address of Next Command	M_1
Product	Address of Multiplier	Address of Multiplicand	Address of Most Significant Part of Product	Address of Least Significant Part of Product	P
Compare	Address of Minuend	Address of Subtrahend	Address of Difference	Address of Next Command if Difference is Non-negative	C
Special Compare	Address of Minuend	Address of Subtrahend	Address of Difference of Absolute Values	Address of Next Command if Difference of Absolute Values is Non-negative	C_1
Extract	Address of Extractor (Determines Digits to be Extracted)	Address of Extractee	Address of Extracted and Shifted Result	If Second Digit of δ is $\begin{cases}0\text{—Shift Left}\\1\text{—Shift Right}\end{cases}$ Other Digits Tell Number of Places to Shift	E
Input	Address of Incoming Information	Drum Address if used		Selects Input Device	I
Special Input	(Incoming Information Goes to Address ϵ)			Selects Input Device	I_1
Output	Address of Outgoing Information	Drum Address if used		Selects Output Device	O

At the beginning of the routine, let the number "49" be placed in address 10. Suppose that address 1 stores the number "1." Place the command

10, 1, 10, 31, C

in address 38. The first time this command immediately following the routine is obeyed, the number in address 10 will be reduced to "48," and the next command will come from address 31. Each time the routine is repeated, the command in address 38 will be executed, with a reduction of the number in address 10 by one. After the routine has been performed 49 times, the number in address 10 will be zero; the 50th time the difference formed by the compare order will be negative, and the next command will come from address 39 (the value of ϵ). Thus, the desired routine will be repeated exactly fifty times.

In the case of the normal addition and subtraction commands, overflow[5] is automatically detected. An extra digit is provided in the arithmetic unit in the most significant end of the A and M[6] registers so that for normal addition and subtraction commands a "1" in this position will cause the next command to come from address δ instead of address ϵ. In the case of the compare command, the proper result of the subtraction is in the A register, but here, as in the addition and subtraction commands, the most significant digit (overflow digit) is not put back into the memory. In the compare command there is no means of detecting overflow.

The extract command provides for obtaining the logical product of two words[7] and shifting the result an arbitrary amount; it may also be used to delete arbitrary parts of a word. Its primary purpose is to assist automatically in fabricating new commands during the computation and to sort out the exponent from the significant figures in floating-point operations.

Special input is the command used to insert information into the computer when it is first put into operation. A word consisting entirely of zeros in the F portion is used to designate special input. Thus, when the memory is cleared, every word is a special input command, and, with these commands, the destination of the incoming data is determined by ϵ. After ϵ has been increased by "1," it determines the source of the next command. Once started, the calculator will count through its complete high-speed memory and read in information from the teletype tape.[8] After the memory has been filled, the ϵ counter steps to zero, and the calculator obeys the command stored in that position.

The input and output commands may specify the magnetic drum as a source or destination of words to be transferred. The drum itself will be about eight inches in diameter and two feet long and will hold about 8,192 words.[9] Initially only one word at a time can be transferred between it and the high-speed memory. The drum will rotate at 3600 revolutions per minute which means that the average access time for a word will be 8 milliseconds. As soon as possible after the computer is put into operation, counting facilities will be added to the control to enable the high-speed memory to be operated in synchronism with the drum. This will make it possible to transfer a whole vector in one revolution of the drum, thus greatly decreasing the average access time.

In general, the drum will serve as an auxiliary storage for numbers, instructions, and function tables. The drum is used instead of extra tape units because it offers better accessibility to information and requires no manual handling. The drum does not hold as much information as a magnetic tape unit, but its size seems adequate for the purposes mentioned above. For greater storage several drums may be operated in synchronism.[10]

The second class of instructions, *control words*, are not directly obeyed by the calculator, nor are they a direct part of the calculation; yet in various ways, they control the course of the computation or enter into the arithmetic-like operations which are performed upon command words. Control words may serve as parameters which determine the number of repetitions of certain routines; they may be the bounds used to stop certain computational processes; or they may serve as factors in the logical products or extraction operations.

The third class of instructions, *code words*,[11] specify (usually in one word)

a whole sequence of procedures for the computer to follow. Thus, one may specify, for example, a scalar multiplication with only one code word. This code word is made up of parameters which specify the common factor (that is, which specify the address in the high-speed memory), the location of the elements of the vector (say, by specifying the address of the first element and the number of terms in the vector), and the location for the result.

When prepared for the computer, a problem consists of a sequence of words called a *main routine*. This main routine is made up of instructions (commands, code words, and control words) together with the numerical constants appropriate to the problem.

The code words in the main routine contain the parameters or addresses necessary to call into action other sequences of instructions, usually called *subroutines*. Subroutines (and code words) exist for such procedures as floating operations, standard iteration formulae, vector operations, integration formulae, and so forth, which are frequently used in the course of doing a computation. A subroutine may itself contain code words which call into use other subroutines. (All of the more frequently-used subroutines will be stored on the magnetic drum, thus being comparatively easily accessible.)

A routine known as the *interpretation routine* keeps track of the place in the main routine, causes segments of the main routine to be brought into the high speed memory, inspects each successive instruction in the main routine to see if it is a command (in which case it is obeyed) or a code word (in which case the interpretation routine extracts an entry[12] from the code word and sends the computer to the appropriate subroutine). A subroutine may itself change the code word being considered by the interpretation routine and, thus, cause the computer to carry out other subordinate activities before going on with the main routine.

Once the appropriate subroutines are established, the task of coding a complex problem is very much simpler. For example, the problem of solving systems of simultaneous linear equations of orders up to 125 by the elimination method has been worked out with the five main subroutines listed in Table 2. The arithmetic operations performed by these subroutines are of

TABLE 2

Routines for Solving Simultaneous Linear Equations

Name	Code	No. of Instructions	Time for Execution in μsecs.	Purpose
Vector Input	α, γ, n, VI	8	$(n + \frac{1}{16})8000$	n words transferred from addresses $\gamma + i$ on drum to addresses $\alpha + i$ of high-speed memory, $i = 0, 1, \cdots, n - 1$.
Vector Output	α, γ, n, VO	9	$(n + \frac{1}{16})8000$	Converse of VI, that is, transfer from high-speed memory to drum.
Floating Operations	α, β, γ, A	88	3,000	Adds, subtracts, multiplies, or divides the operands and puts result in γ.
	α, β, γ, S		3,000	
	α, β, γ, M		6,000	
	α, β, γ, D		14,000	
Vector Constant Product	α, β, γ, n, VC	10	$6500n + 448$	Multiplies word in β by words in $\alpha + i$; answers go into $\gamma + i$, $i = 0, 1, \cdots, n - 1$.
Vector Subtraction	α, β, γ, n, VS	10	$3500n + 448$	Subtracts words in $\beta + i$ from those in $\alpha + i$, answers go into $\gamma + i$, $i = 0, 1, \cdots, n - 1$.

the floating binary type, and each such arithmetic operation is in itself a subroutine. By using code words the coding for this simultaneous linear equation problem is reduced to laying out a sequence of about thirty instructions. It is convenient to have all routines and constants, and two rows of the matrix stored in the high-speed memory; the figure of 125 is based on a 512 word memory. The approximate time required to solve a set of sixty equations under various conditions is given in Table 3. In fixed-point opera-

TABLE 3

Time Required to Solve Sixty Simultaneous Linear Equations[13]

	Fixed Binary Point		Floating Binary Point	
	Synchronized Drum	Non-Synchronized Drum	Synchronized Drum	Non-Synchronized Drum
Computing Time	3.5 min.	3.5 min.	19 min.	19 min.
Transfer Time (to and from the drum)	1.0 min.	30.0 min.	1 min.	30 min.
Totals	4.5 min.	33.5 min.	20 min.	49 min.

tion, a division routine replaces the floating routines of Table 2. The largest pivot may be used in each reduction, and scale factors may have to be introduced. Table 2, Table 3, and some of the routines were worked out by Roselyn Lipkis of the Machine Development Unit at the Institute for Numerical Analysis.

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[1] The Institute is one of four sections of the National Applied Mathematics Laboratories of the National Bureau of Standards. It is located on the campus of the University of California at Los Angeles. The computing machine discussed in this paper is financed by the Air Materiel Command of the United States Air Force.

[2] −N is converted to $2^{48} - N$ or $2^{46} - N$, depending upon whether the size of the memory is 512 or 1024 words, respectively.

[3] These commands are a variation of a set proposed by E. F. Moore while he was working in the National Applied Mathematics Laboratories of the National Bureau of Standards.

[4] For example, in the integration of the exterior ballistic equation, the procedure must change when a shell has completed its flight; or in a square root iteration, the procedure changes when two successive iterants are sufficiently close together.

[5] Addition, subtraction, and compare may produce results which exceed the capacity of the memory cells.

[6] This extra digit in M is needed for the complement process.

[7] Or it may be used for deleting arbitrary parts of a word.

[8] At a later date, magnetic tape may also be used for inserting information into the computer.

[9] Professor P. Morton of the University of California at Berkeley is constructing such a drum.

[10] Professor F. C. Williams of Manchester University, England, has operated a drum in synchronism with a master oscillator.

[11] Code words have been variously termed abbreviated code instructions, quasi-commands, shorthand commands, abbreviated commands, and coded commands. The term "code word" has been selected in preference to these other terms by the author to distinguish more clearly this type of instruction from the explicit commands.

[12] An entry is the address of the command in the subroutine which should be obeyed first.

[13] This table is based on storing three rows of the matrix in the high-speed memory. For more than approximately eighty equations, only two rows can be stored, and input-output is increased by fifty per cent.

DISCUSSIONS

Statistical Treatment of Values of First 2,000 Decimal Digits of e and of π Calculated on the ENIAC

The first 2,000 decimal digits of e and of π were calculated on the ENIAC by Mr. G. REITWIESNER and several members of the ENIAC Branch of the Ballistic Research Laboratories at Aberdeen, Maryland (*MTAC*, v. 4, p. 11–15). A statistical survey of this material has failed to disclose any significant deviations from randomness for π, but it has indicated quite serious ones for e.

Let D_n^i be the number of digits i (where $i = 0, 1, \cdots, 9$) among the first n digits of e or of π. The count begins with the first digit left of the decimal point. If these digits were equidistributed, independent random variables, then the expectation value of each D_n^i (with n fixed and $i = 0, 1, \cdots, 9$) would be $n/10$, and the χ^2 would be

$$a_n = \chi_n^2 = \sum_{i=0}^{9} \left(D_n^i - \frac{n}{10}\right)^2 \Big/ \frac{n}{10}.$$

The system of the D_n^i's (where $i = 0, 1, \cdots, 9$) has 9 degrees of freedom. Therefore let

$$p = P^{(k)}(a) = \frac{1}{2^{\frac{1}{2}k}\Gamma(\frac{1}{2}k)}\int_0^a e^{-\frac{1}{2}x}x^{\frac{1}{2}k-1}dx$$

be the cumulative distribution function of $a = \chi^2$ for k degrees of freedom. Then

$$p_n = P^{(9)}(a_n)$$

is a quantity which would be equidistributed in the interval $[0, 1]$, if the underlying digits were equidistributed independent random variables.

Consider $n = 2000$. In this case, the D_n^i's for e are

(1) 196, 190, 208, 202, 201, 197, 204, 198, 202, 202.

Hence $a_n = \chi_n^2 = 1.11$ and $p_n = .0008$. The D_n^i's for π are

(2) 182, 212, 207, 189, 195, 205, 200, 197, 202, 211.

Hence $a_n = \chi_n^2 = 4.11$ and $p_n = .096$.
The e-value of p_{2000} is thus very conspicuous; it has a significance level of about 1:1250. The π-value of p_{2000} is hardly conspicuous; it has a significance level of about 1:10.

The relevant fact about the distribution (1) appears upon direct inspection. The values lie too close to their expectation value, 200. Indeed their absolute deviations from it are

4, 10, 8, 2, 1, 3, 4, 2, 2, 2,

and hence their mean-square deviation is $22.2 = 4.71^2$, whereas in the random case the expectation value is $180 = 13.4^2$.

In order to see how this peculiar phenomenon develops as n increases to 2000, $a_n = \chi_n^2$ and p_n of e have been determined from D_n^i for the following smaller values of n

n	$a_n = \chi_n^2$	p_n
500	6.72	.33
1000	4.82	.15
1100	5.93	.25
1200	4.03	.093
1300	3.83	.080
1400	4.74	.145
1500	3.69	.070
1600	2.47	.019
1700	3.22	.046
1800	2.85	.031
1900	2.22	.013
2000	1.11	.0008

These numbers show that the abnormally low value of p_n which is so conspicuous at $n = 2000$ does not develop gradually, but makes its appearance quite suddenly around $n = 1900$. Up to that point, p_n oscillates considerably and has a decreasing trend, but at $n = 2000$ there is a sudden dip of quite extraordinary proportions.

Thus something number-theoretically significant may be occurring at about $n = 2000$. A calculation of more digits of e would therefore seem to be indicated. A conversion to a simpler base than 10, say 2, may also disclose some interesting facts.

We wish to thank Miss HOMÉ MCALLISTER of the ENIAC Branch of the Ballistic Research Laboratories for sorting the digital material on which the above analyses are based, and Professor J. W. TUKEY, of Princeton University, for discussions of the subject.

Since the above was written (November 9, 1949), the ENIAC Branch of the Ballistic Research Laboratory very obligingly followed our suggestion and calculated the following 500 additional digits[1] of e. These should replace the last 10 digits of the value of e given in *MTAC*, v. 4, p. 15.

55990	06737	64829	22443	75287	18462	45780	36192	98197	13991
47564	48826	26039	03381	44182	32625	15097	48279	87779	96437
30899	70388	86778	22713	83605	77297	88241	25611	90717	66394
65070	63304	52795	46618	55096	66618	56647	09711	34447	40160
70462	62156	80717	48187	78443	71436	98821	85596	70959	10259
68620	02353	71858	87485	69652	20005	03117	34392	07321	13908
03293	63447	97273	55955	27734	90717	83793	42163	70120	50054
51326	38354	40001	86323	99149	07054	79778	05669	78533	58048
96690	62951	19432	47309	95876	55236	81285	90413	83241	16072
26029	98330	53537	08761	38939	63917	79574	54016	13722	36188

This makes it possible to extend the table of $a_n = \chi_n^2$ and p_n up to $n = 2500$

n	$a_n = \chi_n^2$	p_n
2100	1.94	.0075
2200	2.02	.0088
2300	1.65	.0041
2400	1.70	.0046
2500	1.90	.0070

Thus the values of p_n for $2100 \leqslant n \leqslant 2500$ are still significantly low but higher than the value of p_n at $n = 2000$.

Note that the general size and trend of p_n, as well as its sudden deviation at $n = 2000$, indicate a non random character in the digits of e.

More detailed investigations are in progress and will be reported later.

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Ballistic Research Laboratories G. REITWIESNER

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[1] Both e and $1/e$ were computed somewhat beyond 2500 D and the results checked by actual multiplication.

Notes on Numerical Analysis—2
Note on the Condition of Matrices

1. The object of this note is to establish the following theorem.

THEOREM. *Let A be a real $n \times n$ non-singular matrix and A' be its transpose. Then AA' is more "ill-conditioned" than A.*

This theorem confirms an opinion expressed by Dr. L. FOX[1] based on his practical experience. The term "condition of a matrix" has been used rather vaguely for a long time. The most common measure of the condition of a matrix has been the size of its determinant, ill-conditioned matrices being those with a "small" determinant. With this interpretation imposed, the theorem is clearly correct. More adequate measures of the condition of a matrix have been proposed recently by JOHN VON NEUMANN & H. H. GOLDSTINE[2] and by A. M. TURING.[3] Their definitions concern all matrices, not just the ill-conditioned ones, characterized by very large condition numbers. The following two of these definitions will form a basis for the proof of the above-mentioned theorem:

The P-condition number is $|\lambda_{\max}|/|\lambda_{\min}|$, where $\lambda_{\max}$ and $\lambda_{\min}$ are the characteristic roots of largest and smallest modulus.[2]

The N-condition number is $N(A)N(A^{-1})/n$, where[3]

$$N(A) = (\sum_{i,k} a_{ik}^2)^{\frac{1}{2}}.$$

2. Proof of the theorem in the P case:

Let λ_i be the characteristic roots of A and μ_i those of AA' (which are in general distinct from the squares of the absolute values of λ_i). E. T. BROWNE[4] has shown that

$$\mu_{\min} \leqslant \lambda_i\bar{\lambda}_i \leqslant \mu_{\max}.$$

From this it follows that

$$1 \leqslant \left|\frac{\lambda_{\max}}{\lambda_{\min}}\right| \leqslant \left|\frac{\lambda_{\max}}{\lambda_{\min}}\right|^2 \leqslant \frac{\mu_{\max}}{\mu_{\min}},$$

which implies the required result.

3. Proof of the theorem in the N case:

It is known that $N(A)$ is the square root of the trace of AA' and therefore equal to $(\sum \mu_i)^{\frac{1}{2}}$. The numbers μ_i are all positive since AA' is symmetric and positive definite. Since the characteristic roots of $A'A$ and AA' are the

same and since the characteristic roots of the inverse of a matrix are the reciprocals of those of the original matrix, it follows that

$$N(A^{-1}) = (trA^{-1}(A^{-1})')^{\frac{1}{2}} = (tr(A'A)^{-1})^{\frac{1}{2}} = (\sum \mu_i^{-1})^{\frac{1}{2}}.$$

The N-condition number of A is therefore

$$\frac{1}{n}(\sum \mu_i)^{\frac{1}{2}}(\sum \mu_i^{-1})^{\frac{1}{2}}.$$

In a similar way it can be shown that the N-condition number of AA' is

$$\frac{1}{n}(\sum \mu_i^2)^{\frac{1}{2}}(\sum \mu_i^{-2})^{\frac{1}{2}}.$$

The theorem follows from the inequality

$$\sum \mu_i^2 \sum \mu_i^{-2} \geqslant \sum \mu_i \sum \mu_i^{-1},$$

which is in fact true for all real and positive numbers. (It is, indeed, true when the first power on the right is replaced by an arbitrary power r and the second power on the left by a power $s > r$.) The proof of the inequality is as follows:

$$\begin{aligned}
\sum \mu_i^2 \sum \mu_i^{-2} - \sum \mu_i \sum \mu_i^{-1} \\
&= n + \sum_{i \neq j} \mu_i^2 \mu_j^{-2} - n - \sum_{i \neq j} \mu_i \mu_j^{-1} \\
&= \sum_{i<j} (\mu_i^2 \mu_j^{-2} + \mu_j^2 \mu_i^{-2}) - \sum_{i<j} (\mu_i \mu_j^{-1} + \mu_j \mu_i^{-1}) \\
&= \sum_{i<j} \{(\mu_i \mu_j^{-1} + \mu_j \mu_i^{-1})(\mu_i \mu_j^{-1} + \mu_j \mu_i^{-1} - 1) - 2\} \geqslant 0,
\end{aligned}$$

since

$$\mu_i \mu_j^{-1} + \mu_j \mu_i^{-1} \geqslant 2, \qquad \text{and} \qquad \mu_i \mu_j^{-1} + \mu_j \mu_i^{-1} - 1 \geqslant 1.$$

There is equality if and only if

$$\mu_1 = \mu_2 \cdots = \mu_n.$$

Olga Taussky

NBSMDL

[1] In a course of lectures given at the British Admiralty by himself and D. H. Sadler in 1949.

[2] J. von Neumann & H. H. Goldstine, "Numerical inverting of matrices of high order," Amer. Math. Soc., *Bull.*, v. 53, 1947, p. 1021–1099. (These authors consider symmetric matrices only, but it is reasonable to apply the definition to the general case.)

[3] A. M. Turing, "Rounding-off errors in matrix processes," *Quart. Jn. Mech. Appl. Math.*, v. 1, 1948, p. 287–308.

[4] E. T. Browne, "The characteristic equation of a matrix," Amer. Math. Soc., *Bull.*, v. 34, 1928, p. 363–368.

Bibliography Z–XI

1. E. G. Andrews, "The Bell Computer, Model VI," *Electrical Engineering*, v. 68, 1949, p. 751–756, 7 figs., 5 tables. 22.2 × 29.5 cm.

Controlled from remote stations, this new digital computer of the relay type reduces punched-tape instructions to a minimum. With novel control features similar to those used in recent automatic dial-telephone developments, this "upper-class" computer possesses six "intelligence levels." Sub-

ordinate levels are capable of solving problems such as complex-number multiplication without special guidance.

Author's summary

2. W. R. Ashby, "Design for a brain," *Electronic Engineering*, v. 20, 1948, p. 379–383, figs.

An ideal "thinking machine" must possess negative feedback—or the ability to look after itself cleverly by correcting all deviations from a central, optimal state. The author describes the homeostat, an electro-magnetic machine which is capable of selecting its own arrangement of feedbacks. The designer has merely provided it with plenty of variety so that, if the basic conditions are altered, it can adapt itself. The author points out that the homeostat is still too larval to be a serviceable synthetic "brain."

The merits of the ENIAC, the ACE, and the homeostat are compared in playing the game of chess. The latter needs no detailed instructions—but rather a method by which it is informed of the occurrence of illegal moves and mates. Such a machine, if perfect, could in the opinion of the author eventually play with a subtlety and depth of strategy beyond that of its designer.

There are two apparent disadvantages to this machine. First, the machine will develop a temperament which will probably be manifested in a form too complex for the designer's understanding. More serious in implication is the selfishness of the machine; it will judge the appropriateness of an action by how it is affected by the feedback.

NBSMDL Edith T. Norris

3. Isaac L. Auerbach, J. Presper Eckert, Jr., Robert F. Shaw, & C. Bradford Sheppard, "Mercury delay-line memory using a pulse rate of several megacycles," I.R.E., *Proc.*, v. 37, 1949, p. 855–861, figs.

A mercury delay line memory system for electronic computers, capable of operating at pulse repetition rates of several megacycles per second, has been developed. The high repetition rate results in a saving in space and a reduction in access time.

Numerous improvements in techniques have made the high repetition rate possible. The use of the pulse envelope system of representing data has effectively doubled the possible pulse rate; the use of crystal gating circuits has made possible the control of signals at high pulse rates; and a multi-channel memory using a single pool of mercury has simplified the mechanical construction, reduced the size, and made temperature control much easier.

The memory system described makes possible a significant increase in the over-all speed of an electronic computer.

Author's summary

4. Kay Howard Barney, "The binary quantizer," *Electrical Engineering*, v. 68, 1949, p. 962–967, figs.

The binary quantizer is a new device for translating a time-varying voltage in an analogue computer into a binary number—the size of each being directly proportional to the instantaneous value of the quantity being measured. This makes possible the use of digital techniques in an analogue

computer. Although primarily designed for computing work, this apparatus may find use where quantization of a continuously varying function is desired, as in pulse code modulators, automatic metering devices, and recording machines employing typed or printed numbers.

5. WARREN H. BLISS, "Electronic digital counters," *Electrical Engineering*, v. 68, 1949, p. 309–314, figs.

Recent advancements in all branches of science have created demands for high-speed counting devices. The development of the electronic digital counters, which are used for many purposes in the large computers, has made such fast computations possible. These counters, which use the binary system, are described in detail in this article.

Author's summary

6. D. R. HARTREE, *Calculating Instruments and Machines*, Univ. of Illinois Press, Urbana, Ill., 1949, 138 + ix pages, 68 figs., $4.50. 17 × 25.3 cm.

Professor Hartree has written for the slightly mathematical reader an able summary of existing and projected computers and of some of the mathematical problems which can be solved with their help. His interpretation of the subject is authoritative, as he has personally contributed to many of the phases of the field of computing machinery. Everyone has been intrigued by the Hartree Differential Analyser built of Meccano parts at the University of Manchester in England. More recently Professor Hartree has solved problems on several of the large-scale digital machines, so that he speaks from personal knowledge and experience when he discusses these computers.

In some respects the book has a distinctly English flavor; the terms "instrument" and "machine" are used to denote "analogue" and "digital" computers, respectively. In addition, there are numerous references to developments in England which may be less familiar to American readers than the parallel developments in this country.

After an introductory chapter discussing the distinction between instruments and machines, the author devotes two chapters to the various differential analyzers and to their uses in solving ordinary and partial differential equations. Chapter 4 concludes the section on instruments with a rapid sketch of devices for solving systems of linear algebraic equations, for finding the roots of polynomials, and for integration. (It was noted that no mention was made of the TRAVIS-HART electrical root-finder.)

The section on digital machines opens with a much needed chapter on terminology. A chapter is devoted to BABBAGE's astonishing Analytical Engine of one hundred years ago, and another to existing computers—mechanico-electric, relay, and electronic. Chapter 8 contains schematics and discussions of machines now under construction or in the planning stage.

The book closes with an especially valuable chapter on the methods of numerical analysis, emphasizing the "machine's-eye view" of these methods. The author mentions iterative methods and the solution of algebraic, ordinary and partial differential equations, with useful hints based on his own experience.

Our thanks are due to Professor Hartree for a pleasing and a useful book.

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7. HARRY D. HUSKEY, "The status of high-speed digital computing systems," *Mechanical Engineering*, v. 70, 1948, p. 975–977, bibl.

A brief outline of the history of various computing devices precedes a discussion of technical aspects of many of the current large-scale computers being built all over the world. The author discusses the rôle of computing in science emphasizing the continuing necessity for hand computers and desk machines as well as the high-speed computers. He goes on to point out that the current computers can substitute as a universal model for a large class of model experiments.

EDITH T. NORRIS

NBSMDL

8. T. KILBURN, "The University of Manchester universal high-speed digital computing machine," *Nature*, v. 164, 22 Oct. 1949, p. 684–687.

The author describes the binary high-speed electronic computer being developed under the direction of Professor F. C. WILLIAMS, with the active assistance of the Telecommunications Research Establishment, Malvern.

9. WARREN S. MCCULLOCH, "The brain as a computing machine," *Electrical Engineering*, v. 68, 1949, p. 492–497, bibl.

The author, a medical doctor, employs electrical engineering terminology to show how the brain may be likened to a digital computing machine consisting of ten billion relays called neurons. To carry the analogy further, the performance of the brain is governed by inverse feedback; subsidiary networks secure invariants, or ideas; predictive filters enable us to move toward the place where the object will be when we get there; and complicated servomechanisms enable us to act with facility and precision. Disorders of function are explained in terms of damage to the structure, improper voltage of the relays, and parasitic oscillations.

Author's summary

10. J. HOWARD PARSONS, "Electronic classifying, cataloging, and counting systems," I.R.E., *Proc.*, v. 37, 1949, p. 564–568, figs.

The determination of the distribution of a series of physical events according to magnitude is important in the study of the associated physical laws. Previous methods of determining this magnitude distribution were slow and cumbersome. Three new electronic systems which operate on events at a very high rate have been developed by the author while at Oak Ridge National Laboratory, and these are described.

The analyzers can be used to determine the magnitude distribution of any series of physical events, if the characteristic under observation can be translated into a proportional voltage pulse. Two applications are discussed, and the advantages of the analyzers over other systems are shown.

Author's summary

11. T. PEARCEY, "Modern trends in machine computation," *Australian Journal of Science*, Supplement, v. 10, Feb. 21, 1948, 20 p., 18 figs.

The object of this paper is "to indicate the general structure and organization of high-speed computing machines and the avenues which have been opened up for elaborate high-speed calculations and to suggest the advan-

tages and disadvantages of the various systems available." The author briefly discusses the requirements for an ideal high-speed computer, the basic organic structure, number representation systems, machine components, and simultaneous versus parallel operation. He briefly outlines computer developments which were then current and predicts that in the future these machines may possibly tackle the symbolic procedures of pure mathematics provided the premises are supplied to the machine. The uses of these machines are not limited to the solution of mathematical problems; for example, they could be used to set up efficient, speedy filing systems in industry.

EDITH T. NORRIS

NBSMDL

12. TATSUJIRO SHIMIZU & YOICHI KATAYAMA, "Solution of non-linear equations by punched card methods," *Math. Japonicae*, v. 1, 1948, p. 92–97.

This paper discusses first the solution of simultaneous algebraic equations with particular reference to finding the initial approximation to the roots of the equation using punched card techniques. After the initial difficulty of obtaining the first approximation has been surmounted, closer approximations can easily be found using the NEWTON or other iterative techniques. The author specifically applies his method to the solution of an ordinary differential equation by reducing it to the solution of a system of simultaneous equations.

EDITH T. NORRIS

NBSMDL

13. M. V. WILKES, "Electronic calculating-machine development in Cambridge," *Nature*, v. 164, Oct. 1, 1949, p. 557–561, illustrs.

A very brief description of the EDSAC is presented, and the solution of a typical problem on the machine, i.e., the evaluation of AIRY's integral is discussed.

14. M. V. WILKES, "Programme design for a high-speed automatic calculating machine," *Jn. Sci. Inst. and Phys. in Industry*, v. 26, 1949, p. 217–220.

Problems intended to be solved with the aid of a high-speed digital calculating machine must first be reduced to a series of arithmetical operations. These, together with various auxiliary operations required for such purposes as keeping count of the number of times a particular routine has been repeated, must then be expressed in the code appropriate to the machine. The paper contains an account of some of the details of this process, with special reference to the EDSAC, a high-speed electronic machine in the University Mathematical Laboratory at Cambridge. Simple examples are given. These have been specially designed to illustrate the use of conditional orders and the way in which arithmetical operations may be performed on orders.

Author's summary

15. M. V. WILKES, "Progress in high-speed calculating machine design," *Nature*, v. 164, 27 Aug. 1949, p. 341–345.

An outline is presented of computer developments (chiefly those in England) which were discussed at a conference on high-speed automatic digital calculating machines held in the University Mathematical Laboratory, Cambridge, on June 22–25, 1949, to mark the completion of the EDSAC. [For an account of this conference see *MTAC*, v. 4, p. 51–53.]

16. R. WILSON WILLIAMS, "A survey of some recent advances in computing devices," *Science Progress*, v. 37, 1949, p. 42–52, bibl.

The paper presents a concise survey of recent trends in the design and use of computing devices. Emphasis is laid on those developments involving electronic and electro-mechanical principles.

NEWS

Institute for Numerical Analysis.—A course on electronic computing machines, with special emphasis on the NBS INA Computer, was initiated October 28, 1949, at the Institute for Numerical Analysis. This course, which meets once a week under the direction of Dr. H. D. HUSKEY, is for the purpose of acquainting interested persons with the possibilities and limitations of high-speed digital computers of this type. The course is open to personnel of other governmental and industrial agencies as well as to UCLA and INA staff members.

Institute of Mathematical Statistics.—The forty-first meeting of the Institute of Mathematical Statistics and the twelfth annual meeting was held in New York City on December 27–30, 1949, in conjunction with the meeting of the American Statistical Association, the American Association for the Advancement of Science, the American Mathematical Society, the Econometric Society, the Psychometric Society, the Mathematical Association of America, the Association for Computing Machinery, and the American Psychological Association.

On Friday afternoon, December 30, 1949, a special session of the meeting was devoted to computation at which the following papers were presented:

"Idiosyncrasies of automatically-sequenced digital computing machines" by IDA RHODES, NBS.

"Problem solving on large-scale automatic calculating machines" by W. D. Woo, Harvard University.

"A statistical application of the UNIVAC" by JOHN MAUCHLY, Eckert-Mauchly Computer Corporation.

Discussion by JAMES MCPHERSON, Bureau of the Census, and EMIL SCHELL, Office of the Air Comptroller.

In the first paper, the speaker stressed the fact that the so-called "brain" machines are in reality morons whose every move must be directed. Hence, problems to be put on these machines must be programmed in minute detail; the programmer must foresee every difficulty which might arise in the problem solution. Dr. Woo described the design features of the Mark III calculator, supplementing his discussion with slides. In the UNIVAC discussion, Dr. Mauchly brought out the fact that the UNIVAC was designed primarily for statistical applications; in this machine the emphasis is not on speed for its own sake but rather on rapid input and output speeds needed for Bureau of the Census problems. A detailed discussion of random number generation on the BINAC was also given. During the discussion period following these talks, typical problems to be solved on the UNIVAC were listed, and the time-saving features of this machine were emphasized.

International Business Machines Corporation.—A Seminar on Scientific Computation was held at Endicott, New York, Nov. 16–18, 1949 by the International Business Machines Corporation. The program for the seminar was as follows:

Wednesday Morning, November 16, LEON BRILLOUIN, IBM, Chairman

"The dynamics of nuclear fission" by DAVID L. HILL, Vanderbilt University.

"Monte Carlo calculations" by WILLIAM WOODBURY, Northrop Aviation Co.

"Modifications of the Monte Carlo Method" by HERMAN KAHN, The RAND Corporation.

Wednesday Afternoon, November 16, FRANK HOYT, Argonne National Laboratory, Chairman

"Analyzing exponential decay curves" by ALSTON HOUSEHOLDER, Oak Ridge National Laboratory.

"Eigenvalue problems related to the diffusion equation" by DONALD A. FLANDERS, Argonne National Laboratory and GEORGE SHORTLEY, Operations Research Office, Department of the Army, Johns Hopkins University.

Demonstration of the IBM Card-Programmed Electronic Calculator.

Thursday Morning, November 17, VERNER SCHOMAKER, California Institute of Technology, Chairman.

"Stochastic methods in statistical and quantum mechanics" by GILBERT KING, A. D. Little Company.

"Calculations of resonance energies" by GEORGE KIMBALL, Columbia University.

"Cam design calculations on the card-programmed electronic calculator" by E. A. BARBER, IBM.

Thursday Afternoon, November 17, ARTHUR ROSE, Pennsylvania State College, Chairman

"Distillation theory" by JOHN BOWMAN, Mellon Institute for Industrial Research.

"Calculation of multiple component systems" by STUART R. BRINKLEY, U. S. Bureau of Mines.

Friday Morning, November 18, A. H. TAUB, University of Illinois, Chairman

"The parabolic equation" by L. H. THOMAS, Watson Laboratory.

"Solutions of the wave equation" by PAUL HERGET, Director, Cincinnati Observatory.

"Sampling methods applied to differential equations" by JOHN CURTISS, National Applied Mathematics Laboratories, National Bureau of Standards.

Friday Afternoon, November 18, W. J. ECKERT, Watson Laboratory, Chairman

"On the distribution of KOLMOGOROV's statistic for finite sample size" by Z. W. BIRNBAUM, University of Washington.

"Specialized matrix calculations on the card-programmed electronic calculator" by H. R. J. GROSCH, Watson Laboratory.

Swedish State Board of Computing Machinery. BARK, the Swedish relay machine [*MTAC*, v. 4, p. 52–53] built in Stockholm by C. C. R. A. PALM and collaborators under the Swedish State Board of Computing Machinery is now completed and under trial running. A description of the machine will be given in a future issue of *MTAC*. The mechanical differential analyzer, built in Gothenburg by S. EKELÖF [*MTAC*, v. 3, p. 328] and collaborators has also been completed. Preliminary planning for an electronic computing device is going on under the direction of Ekelöf and Palm.

OTHER AIDS TO COMPUTATION

BIBLIOGRAPHY Z–XI

17. A. E. CARTER & D. H. SADLER, "The application of the National Accounting Machine to the solution of first-order differential equations," *Quart. Jn. Mech. Appl. Math.*, v. 1, 1948, p. 433–441. [*MTAC*, v. 3, p. 548.]

Details are given of the solution on the National machine of the first-order differential equation $y' = f(x, y)$ by the MILNE-STEFFENSEN method.

The formula used is $y_n - y_{n-4} = (3g + 2\delta^2 g + \frac{7}{30}\delta^4 g)_{n-2}$, where $g(x, y) = \frac{4}{3}hf(x, y)$.

The machine is used to build up $(3g + 2\delta^2 g)_{n-2}$ from initial values and $\delta^4 g$ up to $\delta^4 g_{n-3}$, to build up an approximate y_n from y_{n-4}, $(3g + 2\delta^2 g)_{n-2}$, and and extrapolated $\frac{7}{30}\delta^4 g_{n-2}$, and to difference the approximate g_n to $\delta^4 g_{n-2}$. The operator has to provide the extrapolated $\frac{7}{30}\delta^4 g_{n-2}$, to obtain the approximate g_n, and to correct y_n and $\delta^4 g_{n-2}$ by successive approximation. He must further enter these values in the machine at the appropriate times.

The machine prints on a line, n, y_n(approx.), g_{n-1}, g_n(approx.), $\delta^4 g_{n-2}$ (approx.), $\delta^4 g_{n-2}$, $\delta^3 g_{n-3/2}$, $\delta^2 g_{n-1}$, $\delta g_{n-1/2}$, y_{n-3}, and $\frac{7}{30}\delta^4 g_{n-1}$(extrapolated). The second and last of these are corrected by hand.

The next correction term of the series, $-\frac{2}{315}\delta^6 g$, can be applied to y by hand if necessary.

Anyone who wishes to use the Milne-Steffensen method should be warned that it introduces logarithmic rates of increase of error in the series of y_n additional to that, $h\, \partial f/\partial y$, inherent in the differential equation. In this case, the additional logarithmic rates are $\pm \frac{\pi}{2} i + \frac{h}{45}\frac{\partial f}{\partial y}$ and $\pi i - \frac{19}{45} h \frac{\partial f}{\partial y}$. Thus, it may be unwise to use the method to integrate in the direction for which $\partial f/\partial y$ is negative.

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18. A. C. Cook & F. J. Maginniss, "More differential analyzer applications," *Gen. Elec. Rev.*, v. 52, no. 8, 1949, p. 14–20.

This article lists a variety of engineering applications of the G.E. differential analyzer, which has been described in articles by Peterson & Kuehni[1] and by Peterson & Concordia.[2] The applications include electron ballistics, long distance power transmission, bearing design, frequency changers, guided missiles, and control problems. These applications extend those given in a previous paper by Maginniss.[3]

F. J. M.

[1] H. A. Peterson & H. P. Kuehni, "A new differential analyzer," A.I.E.E. *Trans.*, v. 63, 1944, p. 221–228. [*MTAC*, v. 1, p. 430–431.]
[2] H. A. Peterson & C. Concordia, "Analyzers for use in engineering and scientific problems," *Gen. Elec. Rev.*, v. 48, no. 9, 1945, p. 29–37. [*MTAC*, v. 2, p. 55.]
[3] F. J. Maginniss, "Differential analyzer applications," *Gen. Elec. Rev.*, v. 48, no. 5, 1945, p. 54–59. [*MTAC*, v. 1, p. 452–454.]

19. A. B. Macnee, "A High Speed Electronic Differential Analyzer, I.R.E. *Proc.*, v. 37, 1948, p. 1315–1324.

A differential analyzer is described which reproduces its solutions sixty times a second, thus permitting the results to be exhibited on a cathode ray tube. Resetting is accomplished by clamping circuits. Addition and integration are performed in the usual fashion (involving feedback amplifiers). The multiplier uses a cathode ray tube. An electron beam is bent by electrostatic and magnetic methods to yield an effect proportional to the product of two quantities. The resultant deflection is cancelled by a feedback controlled electrostatic deflection, which measures the product. When the feedback is applied to a factor, division is obtained. An arbitrary function is introduced

into the machine by a mask on the face of a cathode ray tube. A feedback arrangement causes the beam to follow the edge of the mask.

An error analysis for differential equations with constant coefficients is also given. The results of applying this differential analyzer to a variety of problems are given including end point boundary problems and the equations of MATHIEU, HILL and VAN DER POL.

F. J. M.

20. J. C. JAEGER & J. D. CLARKE, "A Product Integraph," *Jn. Sci. Instr.*, v. 26, 1949, p. 155–156.

By using mechanical components obtained from surplus anti-aircraft predictors, the authors have built an instrument intermediate in accuracy and cost between Meccano constructions and the well-known large differential analyzers. Two ball-and-disk integrators permit the evaluation of the indefinite integral,

$$\int_0^x f(x)g(x)dx.$$

The input curves $f(x)$ and $g(x)$ are followed manually; the machine is motor-driven in the x direction.

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21. L. PEREK, "Nomograms for computing galactic longitude and latitude, galactic rectangular coordinates and components of the space velocity," Brno, Masarykova Univ., Prirodovĕcká Fakuta, *Spisy*, no. 299, 1947.

The first two nomograms constructed for finding the galactic longitude and latitude of a star are at present of little practical importance, since detailed tables for the conversion of right ascension and declination to galactic longitude and latitude have already been published.[1]

The second set of seven nomographic charts is designed for the calculation of the rectangular components x, y, z of space velocities. The coordinate system is so oriented that the z axis points to the galactic North Pole (RA 12^h40^m, decl. $+28°$), the x axis toward the center of the Galaxy (galact. long. $327°$). The data from which the velocity components are derived are the proper motion components μ_α, μ_δ, the parallax π (or distance r), the radial velocity V, the right ascension α and declination δ of the star. The nomographic solution of this problem is quite accurate enough, as space velocities of stars can generally not be determined with an accuracy greater than 1%. The procedure, however, is by no means simple; it requires eleven nomogram readings of which seven are double constructions. So long as extensive tables are not available these nomograms may, nevertheless, serve a useful purpose.

Three of the charts made for the calculation of velocity components serve also to obtain the rectangular space coordinates of a star (in the same system) from the right ascension, declination, and distance.

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[1] J. OHLSSON, "Tables for the conversion of equatorial coordinates into galactic coordinates," Lund Observatory, *Annals*, v. 3, 1932.

22. ROBERT M. WALKER, "An analogue computer for the solution of linear simultaneous equations," I.R.E., *Proc.*, v. 37, 1949, p. 1467–1473, figs.

Linear simultaneous equations occur frequently in science and in engineering. Their solution by numerical methods is straightforward, but the amount of work required increases rapidly with the number of unknowns. A device is described for the solution of systems of linear simultaneous equations with not more than twelve unknowns. It is an electrical analogue computer which accepts the problem information in digital form from a set of punched cards. This facilitates the preparation, checking, and insertion of the input data and greatly reduces some of the usual liabilities of an analogue device. No special preparation of the problem is required, other than a simple one of scaling the coefficients. Solutions of well-determined problems are easily and rapidly attained and may be refined to any desired accuracy by a simple iteration procedure.

Author's summary

NOTES

112. A COMMITTEE ON FACTOR TABLES.—In September, 1946, the Association Française pour l'avancement des Sciences established a committee consisting of A. GÉRARDIN (France), who to our regret could not participate in the work because of ill health, L. POLETTI (Italy) and the author, for the purpose of extending the factor table. The committee was joined later by Dr. A. GLODEN (Luxemburg). We agreed that only a table practically free from error may have any significant value. Hence, the necessity to check the existing manuscript tables against one another. These tables are: KULIK's famous manuscript,[1] Poletti's table of the 11-th million, GOLUBEV's table of the 11-th and 12-th millions, R. J. PORTER's two tables[2] of the 11-th million. The Carnegie Institution of Washington presented us with two microfilms of the 11-th and 12-th millions of Kulik's manuscript. Poletti's table is in our hands. A request of a photostat of Golubev's table in the Steklov Institute at Moscow finally was denied. Porter's tables are unfortunately in symbols quite different from Kulik's. The committee decided to extend the existing printed table to the 11-th and probably the 12-th million. The necessity of checking Kulik's table against Poletti's made it necessary to get large photos of the relevant part of the microfilm. The Lord Mayor of Luxemburg presented us with large photos of the second half of the 11-th million of the microfilm and the author ordered large photos of the other half. Since Kulik used letters instead of numbers, the author is to transcribe the photos in the interval from 10^7 up to $10^7 + 5\cdot 10^5$, Dr. Gloden in the interval $10^7 + 5\cdot 10^5$ up to $10^7 + 10^6$. Every page of the new manuscript will be checked against Poletti's table. The checking of the manuscript against a third table, which seems necessary is still a problem to the committee.

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[1] See *MTAC*, v. 2, p. 139–140, v. 3, p. 222.
[2] *MTAC*, v. 1, p. 451.

113. Supersonic Flow Calculations.—The following is an account of work done on the ENIAC on supersonic flow past cone cylinders.

Numerical solutions for equations for the flow of a compressible gas at supersonic speeds past a cone cylinder with attached shock wave at various combinations of Mach number and cone semiangle have been computed on the ENIAC [*MTAC*, v. 3, p. 206–207]. The general method of obtaining these solutions will be described first. Using characteristic variables, the equations of irrotational motion for the supersonic flow past a body of revolution may be written as follows:

$$\begin{aligned} Hy_\alpha - (K + R)x_\alpha &= 0 \\ Hy_\beta - (K - R)x_\beta &= 0 \\ Hu_\alpha + (K - R)v_\alpha + (a^2v/y)x_\alpha &= 0 \\ (K - R)u_\beta + Lv_\beta + (a^2v/y)y_\beta &= 0, \end{aligned}$$

where x, y are cylindrical coordinates with the x axis along the axis of the projectile, u, v are reduced velocity components in the direction of the x, y axes, where $u = \bar{u}/c$, $v = \bar{v}/c$, and c is the limiting velocity as the velocity of sound approaches zero, $\bar{u}$, $\bar{v}$ are the original velocity components, $a^2 = \bar{a}^2/c^2$ is the similarly reduced velocity of sound:

$$H = a^2 - u^2, \qquad K = -uv$$
$$L = a^2 - v^2, \qquad R = a(q^2 - a^2)^{\frac{1}{2}}.$$

The surface of the body is assumed to be a stream surface, and with the Rankine-Hugoniot shock wave conditions give us the following boundary conditions, in which the subscripts 1 refer to conditions ahead of the shock and the subscripts 2 refer to conditions immediately behind the shock:

$$y = F(x) \text{ is the contour of the given body, } v = uF'(x),$$
$$\begin{aligned} v_2^2[(\gamma - 1)/(\gamma + 1)q_1 + 2q_1/(\gamma + 1) - u_2] \\ = (q_1 - u_2)^2[u_2 - (\gamma - 1)/(\gamma + 1)q_1] \end{aligned}$$

$dy/dx = (q_1 - u_2)/v_2$, where dy/dx is the slope of the tangent.

$\gamma = 1.4$ is the ratio of the specific heats.

The values of x, y, u, v were computed at the intersections of all characteristics $\alpha =$ const., $\beta =$ const. in the following region. The characteristic $\alpha = 0$ was chosen as the downstream characteristic emanating from the shoulder in Taylor-Maccoll flow about the cone. Along this characteristic the values of x, y, u, v were computed at k intervals $\Delta\beta$ terminating at the shock wave.

The characteristic $\beta = k$ emanates from the intersection of $\alpha = 0$ and the shock wave, and the region under discussion is bounded by $\alpha = 0$, $\beta = k$, and the generating curve of the cylinder. In this region the exact flow is irrotational since the shock wave is straight.

The shoulder of the cone-cylinder is mapped onto the characteristic $\beta = 0$, and Prandtl-Meyer flow is computed along this characteristic, using k points. The generating line of the cylinder is mapped onto the line $\beta = \alpha - k$ in the α, β plane. In the computation of the Taylor-Maccoll flow the partial differential equations were reduced to a set of four simultaneous ordinary differential equations, with

$$\Delta\beta = k(v/u)/[(v/u)\text{cone} - (v/u)\text{shock}].$$

Likewise the equations were reduced to ordinary differential equations for the Prandtl-Meyer flow, with $\Delta\alpha = k(v/u)/(v/u)$cone.

The first computations carried out were for $M = 2.1297$ and a cone cylinder of 20° semi-angle, with $k = 2, 3, 4, 5, 6, 8, 10, 12, 15, 16, 20, 24, 30, 32, 40$. Taylor-Maccoll flow was computed with $k = 960$ and Prandtl-Meyer flow was computed with $k = 16, 32, 64, 128, 256$. The partial derivatives were replaced by difference quotients of first, second and third order approximations, and the results were extrapolated to grid size zero ($k = \infty$). By using the extrapolation formula $z = z_2 + (z_2 - z_1)/3$, where $z = x, y, u, v$, and the subscript 2 indicates that the k for that computation is *twice* the k for the other, it was found that results could be obtained to a much higher accuracy in a short time.

Accordingly, in the next calculation $k = 8$ and 4, and the values of x, y, u, v were calculated at all intersections of the characteristics as well as the values of p/p_1, ρ/ρ_1, T/T_1 at all surface points of the body, where p, ρ, T stand local pressure, density, and temperature, and the subscript 1 indicates free stream values.

The various combinations of cone-angle θ and free stream Mach number M for which computations were made are as follows:

$\theta = 5°$	$M = 1.3, 1.5, 1.7, 2, 3, 4, 5, 7$
$\theta = 9.5°$	$M = 1.3, 1.5, 1.7, 2, 3, 3.8$
$\theta = 10°$	$M = 1.3, 1.5, 1.7, 1.72, 2, 3, 4, 5, 7$
$\theta = 12°$	$M = 1.7$
$\theta = 15°$	$M = 1.3, 1.5, 1.7, 2, 2.13, 2.3, 2.7, 3, 4, 5, 7$
$\theta = 20°$	$M = 1.5, 1.7, 2, 2.13, 3, 4, 5, 7$
$\theta = 25°$	$M = 1.5, 1.7, 2, 2.13, 3, 4, 5, 7$
$\theta = 30°$	$M = 1.7, 2, 2.13, 2.3, 3, 4, 5, 7$
$\theta = 35°$	$M = 2, 2.13, 3, 4, 5, 7$
$\theta = 40°$	$M = 3, 4, 5, 7$
$\theta = 45°$	$M = 3, 4, 5, 7$
$\theta = 50°$	$M = 4, 5, 7$

The results are in some places good to 4 significant figures and it is believed that they are everywhere good to three significant figures.

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114. NBSCL TABLES.—Eleven of these tables have appeared in second editions. We have already referred in *MTAC* to two of these: *Tables of the Bessel Functions $J_0(z)$ and $J_1(z)$ for Complex Arguments*, 1943 (v. 3, p. 25) in 1947; and *Tables of the Exponential Function e^x*, 1939 (v. 3, p .173), in 1947. There were also a second printing of *Table of Circular and Hyperbolic Tangents and Cotangents for Radian Arguments*, 1939 (v. 3, p. 88), in 1947; a second edition of *Tables of Circular and Hyperbolic Sines and Cosines for Radian Arguments*, 1939 (v. 1, p. 45), in 1949; and a second edition of *Tables of Probability Functions*, v. 2, 1942 (v. 1, p. 48), in 1948.

With the approval of the editor of the *Journal of Mathematics and Physics*, the following seven short tables in the MT series were reissued by the Government Printing Office to meet a continuing demand:

MT 19, "On the function $H(m, a, x) = \exp(-ix)F(m + 1 - ia, 2m + 2; ix)$," 1942 (v. 1, p. 156) on 15 June 1949;
MT 20, "Table of integrals $\int_0^x J_0(t)dt$ and $\int_0^x Y_0(t)dt$," 1943 (v. 1, p. 154), 10 Sept. 1948;
MT 21, "Table of $Ji_0(x) = \int_x^\infty J_0(t)dt/t$, and related functions," 1943 (v. 1, p. 155), on 25 Feb. 1949;
MT 22, "Table of coefficients in numerical integration formulae," 1943 (v. 1, p. 157), on 15 Dec. 1949;
MT 23, "Table of Fourier coefficients," 1943 (v. 1, p. 192), on 25 Nov. 1949;
MT 25, "Seven-point Lagrangian integration formulae," 1943, on 1 June 1949;
MT 27, "Table of coefficients for inverse interpolation with central differences," 1943 (v. 1, p. 126, 359), on 1 June 1949.

R. C. A.

115. On Large Primes and Factorizations.—The following results of extensive calculations have been reported.

I. The number $N = 3\cdot 2^{129} + 1$ is composite.
It was found that

$$2^{N-1} \equiv 7949\ 31660\ 65193\ 00342\ 32702\ 86136\ 93493\ 91372 \pmod{N}.$$

If N were a prime, we would have $2^{N-1} \equiv 1 \pmod{N}$.

The number N was suggested as a likely candidate for primality by Mr. Thorold Gosset of Cambridge, England. Thus another attempt to discover a larger prime than $2^{127} - 1$ ends in disappointment.

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II. The number $6^{26} + 1$ is completely factored.

$$6^{26} + 1 = 37\cdot 313\cdot 2341\cdot 629\ 19466\ 95217.$$

The largest factor was proved to be a prime by a method based on the converse of Fermat's theorem, as described in *MTAC*, v. 3, p. 496–497; v. 4, p. 54–55.

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III. (a) The number $N = (2^{89} + 1)/3\cdot 179$ is composite
(b) $3^{36} + 1 = 2\cdot 41\cdot 6481\cdot 28\ 24290\ 05041$
(c) $3^{34} + 1 = 2\cdot 5\cdot 956353\cdot 1743831169$
(d) $2^{61} + 15$ is a prime.

The result (a) follows from the fact that if $y = 2^{89}$,

$$3^{536} - 3^y \equiv 3303\ 41699\ 82572\ 42322\ 50798 \pmod{N}.$$

If N were a prime this remainder would have been zero. The complete factorization of $2^{89} + 1$ would be of interest since it would probably lead to some new multiply perfect numbers.

The results (b) and (c) fill in two blank entries in Cunningham & Woodall, p. 11. The former result was found by the method of *MTAC*, v. 3, p. 96–97. The latter result was found by expressing $(3^{34} + 1)/10$ as the difference of two squares.

The result (d) was another example of the converse of Fermat's theorem. It is easy to see that the 15 integers between the two primes

$$p_1 = 2^{61} - 1 \qquad \text{and} \qquad p_2 = 2^{61} + 15$$

are composite, being multiples of small primes $\leqslant 13$. Hence, p_1 and p_2 are consecutive primes, and constitute the largest pair of consecutive primes known.

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[1] A. J. C. Cunningham & H. J. Woodall, *Factorization of $y^n \pm 1$*, London, 1925.

QUERY

34. Archimides Cattle Problem.—Has any attempt been made to use one of the modern electronic computing devices to get a solution of the famous cattle problem[1] of Archimedes?

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[1] See L. E. Dickson, *History of the Theory of Numbers*, Washington, 1920 and New York, 1934, v. 2, p. 342–345.

QUERIES—REPLIES

44. Tables of $\sin nx/\sin x$ (**Q16**, v. 2, p. 61).—A table of this function "for large integral values of n, say up to 100, and for values of x in radians" would be impossibly large if it were interpolable in x. Millions of entries would be necessary to carry the table as far as $x = 10$ with $8D$. Since isolated values of this function can be obtained with a little trouble from a good table of $\sin x$ it is not surprising that there is no such table in print. For fixed x, however, there are a number of small tables of what is essentially $\sin nx/\sin x$. Two examples may be cited, though they do not correspond to real values of x. For $x = \arccos(-i/2)$ we have

$$\sin nx/\sin x = i^{1-n}F_n,$$

where F_n is the n-th term of the Fibonacci series, which is tabulated as far as $n = 128$ [*MTAC*, v. 2, p. 343]. For $x = \arccos(-3i/\sqrt{2})$ we have

$$\sin nx/\sin x = (i\sqrt{2})^{1-n}(2^n - 1)$$

the values of which can be found easily from a table of powers of 2.

As is well known, $U_n = \sin nx/\sin x$ is a special kind of Lucas[1] function and can be computed recurrently by the formula

$$U_{n+1} = 2\cos x U_n - U_{n-1}.$$

D. H. L.

[1] É. Lucas, *Théorie des Nombres*, Paris 1890, p. 319.

CORRIGENDA

V. 3, p. 362, l. −1 *for* RMT **593** *read* RMT **592**.
V. 3, p. 554, l. 3, *for* 4–17, *read* 3–16.
V. 3, p. 559, l. 4, *for* 454 *read* 554.
V. 3, p. 562, l. 30, *for* Nicholas De Cusa *read* Nicholas de Cusa.
V. 3, p. 563, l. 6, *for* fifteenth *read* thirteenth.
Nos. 28, 29, cover 3, interchange lines J and I.